Algebra II
Fall 2017

Assignment 8.1
Due October 25

Exercise 1. Let F be a field, V be a vector space over F and $\text{End}(V)$ the set of all vector space endomorphisms over V. Show that $\text{End}(V)$ is a ring with unity under the operations of point-wise addition and composition. We will be considering it as an R-module over itself.

Exercise 2. Let F be a field and V be a vector space over F. It is not hard to show that every element of $\text{End}(V)$ is specified and uniquely defined by its action on a basis of V. Let $V = F[x]$ with basis $\{1, x, x^2, \ldots\}$ and define $A, B \in R = \text{End}(F[x])$ by declaring

$$A(x^{2n}) = x^n, \;\; A(x^{2n+1}) = 0, \;\; B(x^{2n}) = 0, \;\; B(x^{2n+1}) = x^n \quad \text{for } n \in \mathbb{N}_0.$$

Let $S, T \in R$ and suppose that $SA + TB = 0$. By evaluating this linear combination at x^{2n} and x^{2n+1}, conclude that $S = T = 0$. Hence A and B are R-linearly independent elements of R.

Exercise 3. Continuing the exercise above, define $S, T \in R$ through

$$S(x^n) = x^{2n}, \;\; T(x^n) = x^{2n+1} \quad \text{for } n \in \mathbb{N}_0.$$

By evaluating at the even and odd powers of x separately, show that $SA + TB = I$, the identity endomorphism. Use this to conclude that, as an R-module, $\langle A, B \rangle = R$.

Remark. The preceding exercises show that $\{A, B\}$ is an R-module basis for R. But clearly so is $\{I\}$, since $TI = T$ for all T. Hence R is a free R-module, but the size of its bases is not unique. In fact, it's not difficult to modify the preceding exercises to produce bases of size m for any $m \in \mathbb{N}$.