PUTNAM SEMINAR
FALL 2018

QUIZ 11
DUE NOVEMBER 7

Name:______________________________

Start Time:________________________

End Time:_________________________

Problem 1. Let $*$ be a commutative and associative binary operation on a set S. Assume that for every x and y in S, there exists z in S such that $x * z = y$ (this z may depend on x and y). Show that if a, b, c are in S and $a * c = b * c$, then $a = b$.

Problem 2. In the additive group of ordered pairs of integers (with addition defined componentwise), consider the subgroup H generated by the three elements

$$(3, 8), \quad (4, -1), \quad (5, 4).$$

Then H has another set of generators of the form

$$(1, b), \quad (0, a),$$

for some integers a, b with $a > 0$. Find a.

Problem 3. Let G be a group with identity e and $\phi : G \to G$ a function such that

$$\phi(g_1)\phi(g_2)\phi(g_3) = \phi(h_1)\phi(h_2)\phi(h_3)$$

whenever $g_1 g_2 g_3 = e = h_1 h_2 h_3$. Prove that there exists an element $a \in G$ such that $\psi(x) = a\phi(x)$ is a homomorphism (i.e. $\psi(xy) = \psi(x)\psi(y)$ for all $x, y \in G$).

Problem 4. Let G be a finite set of real $n \times n$ matrices $\{M_i\}$, $1 \leq i \leq r$, which form a group under matrix multiplication. Suppose that $\sum_{i=1}^{r} \operatorname{tr}(M_i) = 0$, where $\operatorname{tr}(A)$ denotes the trace of the matrix A. Prove that $\sum_{i=1}^{r} M_i$ is the $n \times n$ zero matrix.

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