Introduction to Abstract Mathematics — Assignment 10.1/2

Fall 2018 — Due November 7

Exercise 1. Prove that $|\mathbb{Z}| = |\mathbb{N}|$.

Exercise 2. Let A be a set.

a. Prove that if $A \subset \mathbb{N}$ is infinite, $|A| = |\mathbb{N}|$.

b. Prove the following weak version of the Schroeder-Bernstein Theorem. If $|A| \leq |\mathbb{N}| \leq |A|$, then $|A| = |\mathbb{N}|$.

Exercise 3.

a. Show that $|(-1, 1)| = |\mathbb{R}|$. [*Suggestion:* Use an example from class.]

b. Prove that the map $x \mapsto x - 1/x$ provides a bijection between $(0, \infty)$ and $\mathbb{R}$.

c. Let $a, b, c, d \in \mathbb{R}$. If $a < b$ and $c < d$, find a bijection from (a, b) to (c, d).

d. Conclude that all open intervals in $\mathbb{R}$ (finite, infinite, or half-infinite) have the same cardinality.

Exercise 4. Let S be an infinite set and $x \in S$.

a. Show that there exists an injection $f : \mathbb{N} \to S$ so that $f(1) = x$. [*Suggestion:* By one of our characterizations of infinite sets, there is a surjection $S \to \mathbb{N}$. Compose with an appropriately chosen bijection $\mathbb{N} \to \mathbb{N}$ to arrange it so that $x \mapsto 1$. Quote a theorem about surjections to complete the proof.]

b. Use part **a** to show that $|S \setminus \{x\}| = |S|$.

c. Use part **b** to prove that if $x_1, x_2, \ldots, x_n \in S$, then $|S \setminus \{x_1, x_2, \ldots, x_n\}| = |S|$. [*Suggestion:* Remove one element of S at a time. Given that this changes the set S at each stage, how do you know **b** still holds?]

d. Prove that $|[0, 1]| = |(0, 1]| = |[0, 1)| = |(0, 1)|$. The proof of this counterintuitive result eluded me as an undergraduate.

Exercise 5. Let S be a set, $\mathcal{P}^0(S) = S$, and for $n \geq 1$ set

$$\mathcal{P}^n(S) = \mathcal{P}(\mathcal{P}^{n-1}(S)).$$

Prove that

$$|\mathcal{P}^m(S)| < \left| \bigcup_{n \in \mathbb{N}_0} \mathcal{P}^n(S) \right|$$

for all $m \in \mathbb{N}_0$. Interpret this result in the case that $S = \mathbb{N}$. [*Suggestion:* Show directly that $\leq$ holds, then argue by contradiction to show that equality does not.]