Introduction to Abstract Mathematics — Assignment 12.2
Fall 2018

Exercise 1. The *FIbonacci numbers* are the terms in the sequence defined as follows:

$$F_0 = 0, \;\; F_1 = 1, \;\; F_{n+2} = F_{n+1} + F_n \quad \text{for } n \geq 0.$$

Conjecture and prove a formula for $F_0 + F_1 + F_2 + \cdots + F_n$.

Exercise 2. For $n \in \mathbb{N}$, the nth *harmonic number* is $H_n = \displaystyle\sum_{k=1}^{n} \frac{1}{k}$. Prove that for all $n \geq 0$, $H_{2^n} \geq 1 + \frac{n}{2}$.

Exercise 3. Let $n \in \mathbb{N}$, $n \geq 2$. Use induction to prove that if $A_1, A_2, \ldots, A_n$ are countable, then the set

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \,|\, a_i \in A_i \text{ for all } i\}$$

is countable.

Exercise 4. What's wrong with the following "proof" that all M&Ms have the same color?

Proof. It suffices to prove that all the M&Ms in any finite set S have the same color. For this we induct on $n = |S|$. When $n = 1$ the result is clear, since one M&M has only a single color. Now let $n \in \mathbb{N}$ and suppose the result is true for all sets of n M&Ms. Let $S = \{m_1, m_2, \ldots, m_{n+1}\}$ be a set of $n + 1$ M&Ms. Because they each only have n M&Ms, the inductive hypothesis applies to the sets $S_1 = \{m_1, \ldots, m_n\}$ and $S_2 = \{m_2, \ldots, m_n\}$. That is, the M&Ms in S_1 all have a common color, and the M&Ms in S_2 all have a common color. Since the two sets share the element m_n, the two common colors must actually be the same. Hence, all the elements in $S = S_1 \cup S_2$ have a single color. The result now follows by induction. $\square$