Introduction to Abstract Mathematics — Assignment 7

Fall 2018 — Due October 17

Exercise 1.

- **a.** Prove that the cube of any integer is within 1 of a multiple of 9.
- **b.** Prove that the fourth power of any integer is either a multiple of 5 or 1 more than a multiple of 5.

Exercise 2. Let $a, b, c \in \mathbb{Z}$, with a and b not both zero. Prove that if $c|a$ and $c|b$, then $c|\gcd(a,b)$. [*Suggestion:* Use Bézout's Lemma.]

Remark. If we let $S = \{c \in \mathbb{N} \mid c|a \text{ and } c|b\}$, then the result of this exercise can be used to show that

$$\gcd(a,b) = \max S = \operatorname{lcm} S.$$

This is worth noting since although it's immediate that $c \leq \operatorname{lcm} S$ for all $c \in S$, it's not clear that $\operatorname{lcm} S \in S$.

Exercise 3. Let $a, b, k \in \mathbb{Z}$ with a and b not both zero and $k > 0$. Prove that $\gcd(ka, kb) = k\gcd(a,b)$. [*Suggestion:* Use Bézout's Lemma to show that the LHS and RHS divide one another.

Remark. This can be used to prove that if $d = \gcd(a,b)$, $a = da'$ and $b = db'$, then $\gcd(a', b') = 1$, which is equivalent to the statement that it's always possible to write a fraction in "lowest terms."

Exercise 4. Let $a, b \in \mathbb{Z}$ be nonzero and let $S = \{c \in \mathbb{N} \mid a|c \text{ and } b|c\}$, the set of positive *common multiples* of a and b. Use the Well-Ordering Principle to show that S has a least element, m, and the Division Algorithm to show that m divides every element of S. The integer m is called the *least common multiple* of a and b.