Putnam Seminar
Fall 2019

Quiz 1
Due September 4

Name:______________________________

Start Time:_________________________

End Time:__________________________

Problem 1. A function f is defined for all positive integers and satisfies

$$f(1) = 2019 \text{ and } f(1) + f(2) + \cdots + f(n) = n^2 f(n).$$

Compute the exact value of $f(2019)$.

Problem 2. Let $Q_0(x) = 1$, $Q_1(x) = x$, and

$$Q_n(x) = \frac{(Q_{n-1}(x))^2 - 1}{Q_{n-2}(x)}$$

for all $n \geq 2$. Show that, whenever n is a positive integer, $Q_n(x)$ is equal to a polynomial with integer coefficients.

Problem 3. Let A be the $n \times n$ matrix whose entry in the i-th row and j-th column is

$$\frac{1}{\min(i, j)}$$

for $1 \leq i, j \leq n$. Compute $\det(A)$.

Problem 4. Show that every positive rational number can be written as a quotient of products of factorials of (not necessarily distinct) primes. For example,

$$\frac{10}{9} = \frac{2! \cdot 5!}{3! \cdot 3! \cdot 3!}.$$

Problem 1. A function f is defined for all positive integers and satisfies

$$f(1) = 2019 \text{ and } f(1) + f(2) + \cdots + f(n) = n^2 f(n).$$

Compute the exact value of $f(2019)$.

Problem 2. Let $Q_0(x) = 1$, $Q_1(x) = x$, and

$$Q_n(x) = \frac{(Q_{n-1}(x))^2 - 1}{Q_{n-2}(x)}$$

for all $n \geq 2$. Show that, whenever n is a positive integer, $Q_n(x)$ is equal to a polynomial with integer coefficients.

Problem 3. Let A be the $n \times n$ matrix whose entry in the i-th row and j-th column is

$$\frac{1}{\min(i, j)}$$

for $1 \leq i, j \leq n$. Compute $\det(A)$.

Problem 4. Show that every positive rational number can be written as a quotient of products of factorials of (not necessarily distinct) primes. For example,

$$\frac{10}{9} = \frac{2! \cdot 5!}{3! \cdot 3! \cdot 3!}.$$