Putnam Seminar
Fall 2019

Quiz 13
Due December 4

Name:______________________________

Start Time:__________________________

End Time:___________________________

Problem 1. Let n be an even positive integer. Write the numbers $1, 2, \ldots, n^2$ in the squares of an $n \times n$ grid so that the k-th row, from left to right, is

$$(k-1)n+1, (k-1)n+2, \ldots, (k-1)n+n.$$

Color the squares of the grid so that half of the squares in each row and in each column are red and the other half are black (a checkerboard coloring is one possibility). Prove that for each coloring, the sum of the numbers on the red squares is equal to the sum of the numbers on the black squares.

Problem 2. Let $a_0 = 5/2$ and $a_k = a_{k-1}^2 - 2$ for $k \geq 1$. Compute $\displaystyle\prod_{k=0}^{\infty}\left(1 - \frac{1}{a_k}\right)$ in closed form.

Problem 3. Let f be a function on $[0, \infty)$, differentiable and satisfying $f'(x) = -3f(x) + 6f(2x)$ for $x > 0$. Assume that $|f(x)| \leq e^{-\sqrt{x}}$ for $x \geq 0$ (so that $f(x)$ tends rapidly to 0 as x increases). For n a non-negative integer, define

$$\mu_n = \int_0^\infty x^n f(x)\, dx$$

(sometimes called the nth moment of f).

a) Express μ_n in terms of μ_0.

b) Prove that the sequence $\{\mu_n \frac{3^n}{n!}\}$ always converges, and that the limit is 0 only if $\mu_0 = 0$.

Problem 4. Suppose that $f(x) = \sum_{i=0}^{\infty} c_i x^i$ is a power series for which each coefficient c_i is 0 or 1. Show that if $f(2/3) = 3/2$, then $f(1/2)$ must be irrational.

Problem 1.

Problem 2.

Problem 3.

Problem 4.