PUTNAM SEMINAR
FALL 2019

QUIZ 5
DUE OCTOBER 2

Name:______________________________

Start Time:_________________________

End Time:__________________________

Problem 1. Suppose f and g are non-constant, differentiable, real-valued functions defined on $(-\infty, \infty)$. Furthermore, suppose that for each pair of real numbers x and y,

$$f(x+y) = f(x)f(y) - g(x)g(y),$$
$$g(x+y) = f(x)g(y) + g(x)f(y).$$

If $f'(0) = 0$, prove that $(f(x))^2 + (g(x))^2 = 1$ for all x.

Problem 2. Solve the equations

$$\frac{dy}{dx} = z(y+z)^n, \quad \frac{dz}{dx} = y(y+z)^n,$$

given the initial conditions $y = 1$ and $z = 0$ when $x = 0$.

Problem 3. Suppose that the function $h : \mathbb{R}^2 \to \mathbb{R}$ has continuous partial derivatives and satisfies the equation

$$h(x,y) = a\frac{\partial h}{\partial x}(x,y) + b\frac{\partial h}{\partial y}(x,y)$$

for some constants a, b. Prove that if there is a constant M such that $|h(x,y)| \leq M$ for all $(x,y) \in \mathbb{R}^2$, then h is identically zero.

Problem 4. Let f be a twice-differentiable real-valued function satisfying

$$f(x) + f''(x) = -xg(x)f'(x),$$

where $g(x) \geq 0$ for all real x. Prove that $|f(x)|$ is bounded.

Problem 1. Suppose f and g are non-constant, differentiable, real-valued functions defined on $(-\infty, \infty)$. Furthermore, suppose that for each pair of real numbers x and y,

$$f(x+y) = f(x)f(y) - g(x)g(y),$$
$$g(x+y) = f(x)g(y) + g(x)f(y).$$

If $f'(0) = 0$, prove that $(f(x))^2 + (g(x))^2 = 1$ for all x.

Problem 2. Solve the equations

$$\frac{dy}{dx} = z(y+z)^n, \quad \frac{dz}{dx} = y(y+z)^n,$$

given the initial conditions $y = 1$ and $z = 0$ when $x = 0$.

Problem 3. Suppose that the function $h : \mathbb{R}^2 \to \mathbb{R}$ has continuous partial derivatives and satisfies the equation

$$h(x, y) = a\frac{\partial h}{\partial x}(x, y) + b\frac{\partial h}{\partial y}(x, y)$$

for some constants a, b. Prove that if there is a constant M such that $|h(x, y)| \leq M$ for all $(x, y) \in \mathbb{R}^2$, then h is identically zero.

Problem 4. Let f be a twice-differentiable real-valued function satisfying

$$f(x) + f''(x) = -xg(x)f'(x),$$

where $g(x) \geq 0$ for all real x. Prove that $|f(x)|$ is bounded.