Example 1. Find an equation for the plane containing the points $A = (9, -8, 8), B = (10, -2, 10)$ and $C = (8, -4, 7)$.

Solution. We need a point on the plane and a vector normal to the plane. We have no shortage of points. As far as the normal goes, because the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ both lie in the plane, their cross product must necessarily be perpendicular to the plane. We may therefore take $\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC}$. We have

$$\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \langle 1, 6, 2\rangle \times \langle -1, 4, -1\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 6 & 2 \\ -1 & 4 & -1 \end{vmatrix} = -14\mathbf{i} - \mathbf{j} + 10\mathbf{k}.$$

Taking A as our base point, we find that an equation for the plane in question is

$$-14(x - 9) - (y + 8) + 10(z - 8) = 0,$$

which, after simplification, is equivalent to

$$-14x - y + 10z = -38.$$

□

Example 2. Show that the planes $2x - 5y + 9z = 6$ and $4x - 10y + 11z = 0$ are not parallel, and find parametric equations for their line of intersection.

Solution. The components of the normal vector to a plane are the coefficients of x, y and z in its equation. So we find that the normal vectors to the two planes in question are

$$\mathbf{n}_1 = \langle 2, -5, 9\rangle,$$
$$\mathbf{n}_2 = \langle 4, -10, 11\rangle.$$

In order for the two planes to be parallel, their normal vectors must have the same (or opposite) directions. In terms of vector algebra, this means that they must be scalar multiples of one another. This is clearly not the case, since in order to scale $\mathbf{n}_1$ into $\mathbf{n}_2$ we'd have to multiply by 2 so that the first components would match, but then the third components would differ. Therefore the planes cannot be parallel.

If a line L lies in a plane P, then the direction of L must be orthogonal to the normal to P, because the normal is orthogonal to all vectors lying in P by definition. Since the line of intersection we seek lies in both of the given planes simultaneously, this means that its direction must be orthogonal to *both* $\mathbf{n}_1$ and $\mathbf{n}_2$. But the only direction orthogonal to both of these vectors is determined by $\mathbf{n}_1 \times \mathbf{n}_2$. That is, the direction of the line of intersection is given by

$$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -5 & 9 \\ 4 & -10 & 11 \end{vmatrix} = 35\mathbf{i} + 14\mathbf{j} = 7(5\mathbf{i} + 2\mathbf{j}).$$

Because it has the same direction but smaller components, we will choose to use the vector $5\mathbf{i} + 2\mathbf{j}$ instead. We mention that because the cross product is not the zero vector, this also proves that $\mathbf{n}_1$ and $\mathbf{n}_2$ are not parallel.

To find a base point for our line, we need any solution of the system

$$\begin{aligned} 2x - 5y + 9z &= 6, \\ 4x - 10y + 11z &= 0. \end{aligned}$$

Because no line is parallel to all of the coordinate planes, we can always take one of the coordinates of our point to be zero, in order to make solving the system easier. In "most" situations, the choice of which coordinate coordinate is immaterial. But in our case it is not. The direction vector $5\mathbf{i} + 2\mathbf{j}$ lacks a z-component, making our line parallel to the xy-plane. However, it will still hit the xz-plane and the yz-plane, so we are free to choose either $y = 0$ or $x = 0$. Making the former choice, our system becomes

$$\begin{aligned} 2x + 9z &= 6, \\ 4x + 11z &= 0. \end{aligned}$$

Gaussian elimination (or matrix inversion) shows that the unique solution to this system is $x = \frac{33}{7}$, $y = \frac{-12}{7}$. Thus, the point

$$\left(\frac{33}{7}, 0, \frac{-12}{7}\right)$$

lies on the line of intersection.

Putting the point and direction together yields the parametric equations

$$\begin{aligned} x &= \frac{33}{7} + 5t, \\ y &= 2t, \\ z &= \frac{-12}{7}, \end{aligned}$$

for the line of intersection. $\square$

Example 3. Show that the planes $3x - 2y + z = 12$ and $x + 3y - 5z = 7$ are not parallel, and find the acute angle between them.

Solution. As above, it is apparent that the normals to the two planes are

$$\begin{aligned} \mathbf{n}_1 &= \langle 3, -2, 1 \rangle, \\ \mathbf{n}_2 &= \langle 1, 3, -5 \rangle. \end{aligned}$$

Because the normal vectors make the same angle with both planes, namely 90°, the angle between the planes is equal to either the angle between the normals, or its supplement. The angle between the normals satisfies

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| \cdot |\mathbf{n}_2|} = \frac{3 - 6 - 5}{\sqrt{(3^2 + 2^2 + 1^2)(1^2 + 3^2 + 5^2)}} = \frac{-8}{\sqrt{14 \cdot 35}} = \frac{-8}{7\sqrt{10}}.$$

Since the cosine is negative, we know that $\frac{\pi}{2} < \theta \leq \pi$. In particular, θ is obtuse, so the angle we really want is

$$\pi - \theta = \pi - \arccos\frac{-8}{7\sqrt{10}} \approx 68.81°.$$

$\square$