Example 1. Let $\mathbf{a} = \langle -2, 0, 3\rangle$ and $\mathbf{b} = \langle 1, 4, 1\rangle$. Compute the scalar and vector projections of $\mathbf{a}$ on $\mathbf{b}$ and vice versa.

Solution. According to the formulae derived in class, the scalar projections are

$$\begin{aligned} \mathrm{comp}_{\mathbf{b}}(\mathbf{a}) &= \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|} = \frac{-2+0+3}{\sqrt{1^2+4^2+1^2}} = \frac{1}{3\sqrt{2}}, \\ \mathrm{comp}_{\mathbf{a}}(\mathbf{b}) &= \frac{\mathbf{b}\cdot\mathbf{a}}{|\mathbf{a}|} = \frac{-2+0+3}{\sqrt{2^2+0^2+3^2}} = \frac{1}{\sqrt{13}}, \end{aligned}$$

and the vector projections are

$$\begin{aligned} \mathrm{proj}_{\mathbf{b}}(\mathbf{a}) &= \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\right)\mathbf{b} = \frac{1}{18}\langle 1,4,1\rangle = \left\langle \frac{1}{18}, \frac{2}{9}, \frac{1}{18}\right\rangle, \\ \mathrm{proj}_{\mathbf{a}}(\mathbf{b}) &= \left(\frac{\mathbf{b}\cdot\mathbf{a}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a} = \frac{1}{13}\langle -2,0,3\rangle = \left\langle \frac{-2}{13}, 0, \frac{3}{13}\right\rangle. \end{aligned}$$

Note that interchanging the roles of $\mathbf{a}$ and $\mathbf{b}$ gives different values for the projections. □

Example 2. If a force $\mathbf{F}$ moves an object through a displacement $\mathbf{d}$, the work done is

$$W = \mathrm{proj}_{\mathbf{F}}(\mathbf{d}) \cdot |\mathbf{d}| = \mathbf{F}\cdot\mathbf{d}.$$

So one might say that the dot product of two (arbitrary) vectors measures the "energy" of their configuration.

Example 3. A sled is pulled 80 ft by a 30 lb force directed at an angle 40° above horizontal. How much work is done by this force?

Solution. Because the angle between the force and the displacement is 40°, we find that

$$W = \mathbf{F}\cdot\mathbf{d} = |\mathbf{F}|\cdot|\mathbf{d}|\cos 40^\circ = 80\cdot 30\cos 40^\circ \text{ ft lb} = 2400\cos 40^\circ \text{ ft lb},$$

which is approximately 1838.5 ft lb. □