Modern Algebra II
Fall 2019

Assignment 10.1
Due November 20

Exercise 1. Let K/F be fields and suppose $S, T \subset K$. Prove that $F(S)(T) = F(S \cup T)$.

Exercise 2. Let K/F be fields. Suppose that $\alpha \in K$ is one root of an irreducible quadratic polynomial $f(X) \in F[X]$. Prove that $F(\alpha) = \{a + b\alpha \mid a, b \in F\}$.

Exercise 3. Let $p, q \in \mathbb{Z}$ be distinct prime numbers.

a. Prove that $X^2 - q$ is irreducible over $\mathbb{Q}(\sqrt{p}) = \{a + b\sqrt{p} \mid a, b \in \mathbb{Q}\}$.

b. Use part **a** and Exercise 2 to conclude that

$$\mathbb{Q}(\sqrt{p}, \sqrt{q}) = \{a + b\sqrt{p} + c\sqrt{q} + d\sqrt{pq} \mid a, b, c, d \in \mathbb{Q}\}.$$

c. Show that $\mathbb{Q}(\sqrt{p}, \sqrt{q}) = \mathbb{Q}(\sqrt{p} + \sqrt{q})$.

Exercise 4. Let F be a field. The quotient field of the polynomial ring $F[X]$ is denoted $F(X)$ and is called the *field of rational functions* in X. Let $n \in \mathbb{N}$ consider the polynomial $f(X, T) = T^n - X \in F[X, T]$.

a. Show that f is irreducible as a polynomial in X over the field $F(T)$ of rational functions in T.

b. Use (a corollary to) Gauss' lemma to conclude that f is irreducible in $F[X, T]$.

c. Use similar reasoning to conclude that f is irreducible as a polynomial in T over the field $F(X)$.

d. Conclude that for $n \geq 2$, $X^{1/n}$ is "irrational," i.e. $X^{1/n} \notin F(X)$.

Exercise 5. Let $p \in \mathbb{Z}$ be a prime number and let $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ be the field with p elements. Show that the extension $\mathbb{F}_p(X^{1/p})$ of $\mathbb{F}_p(X)$ contains every root of $T^p - X \in \mathbb{F}_p(X)[T]$.