MODERN ALGEBRA II
FALL 2019

ASSIGNMENT 10.2
DUE NOVEMBER 20

Exercise 1. Let F be a field.

a. If $a \in F$ and $a \neq \square$, show that $X^2 - a \in F[X]$ is irreducible.

b. Let $a \in F$ with $a \neq \square$, and let $\sqrt{a}$ denote a fixed root of $X^2 - a$ in some extension K/F. Prove that

$$k = \left\{ x + y\sqrt{a} \,\middle|\, x, y \in F \right\}$$

is a subfield of K. Conclude that $k = F\left(\sqrt{a}\right)$.

c. Let $a, b \in F$ with $a, b \neq \square$. Let K/F be an extension containing both $\sqrt{a}$ and $\sqrt{b}$. Prove that $F\left(\sqrt{a}\right) = F\big(\sqrt{b}\big)$ if and only if there is a $c \in F^\times$ so that $a = bc^2$.

Exercise 2. Let F be a field and let $f_1, \ldots, f_n \in F[X]$. Prove that a field K/F is a splitting field over F for the set $\{f_1, f_2, \ldots, f_n\}$ if and only if K is a splitting field over F for the single polynomial $f_1 f_2 \cdots f_n$.

Exercise 3. Lek $K/k/F$ be fields and let $f \in F[X]$. Prove that if K is a splitting field of f over F, then K is a splitting field of f over k.