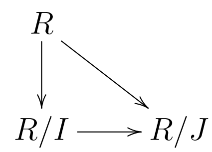
Modern Algebra II
Fall 2019

Assignment 3.1
Due September 18

Exercise 1. Let $I \subset J$ be ideals in a ring R. Prove that the rule $a + I \mapsto a + J$ yields a well-defined homomorphism $R/I \to R/J$, and that the diagram



commutes, where the vertical and diagonal arrows are the canonical surjections.

Exercise 2. Let I be an ideal in a commutative ring R. The *radical* of I is the set

$$\sqrt{I} = \{a \in R \,|\, a^n \in I \text{ for some } n \in \mathbb{N}\},$$

which consists of all the elements in R that are nth roots of some element in I, for all $n \geq 1$.

a. Prove that $\sqrt{I}$ is an ideal in R containing I.

b. Prove that $\sqrt{\sqrt{I}} = \sqrt{I}$. Conclude that $R/\sqrt{I}$ has no nonzero nilpotent elements.

c. Which elements of R are in $\sqrt{0}$?

Exercise 3. Let I be a nonzero proper ideal in $\mathbb{Z}$. Prove that there is a constant $C(I)$ so that if $I = I_1 \subset I_2 \subset \cdots \subset I_n$ is a strictly increasing sequence of proper ideals in $\mathbb{Z}$, then $n \leq C(I)$.