Modern Algebra II
Fall 2019

Assignment 3.2
Due September 18

Exercise 1. Let $\varphi : R \to R'$ be a homomorphism of rings.

a. If J is an ideal in R', prove that $\varphi^{-1}(J)$ is an ideal in R.

b. If I is an ideal in R and φ is surjective, prove that $\varphi(I)$ is an ideal in R'.

Exercise 2. A collection $\mathcal{C}$ of ideals in a ring R is called a *chain* if for all $I, J \in \mathcal{C}$, either $I \subset J$ or $J \subset I$. Let $\mathcal{C}$ be a chain of ideals in R.

a. Prove that $J = \bigcup_{I \in \mathcal{C}} I$ is an ideal in R.

b. If every ideal in $\mathcal{C}$ is proper, prove that J is, too.

Exercise 3. Let F be a field. A surjective homomorphism $\nu : (F^\times, \cdot) \to (\mathbb{Z}, +)$ is called a *discrete valuation* if $\nu(a+b) \geq \min\{\nu(a), \nu(b)\}$ for all $a, b \in F^\times$ (with $a \neq -b$). Note that if we set $\nu(0) = \infty$, then the multiplicative and additive properties of ν extend to all of F.

a. Prove that $R = \{a \in F \mid \nu(a) \geq 0\}$ is a subring of F, the *valuation ring*.

b. Show that $R^\times = \ker \nu$.

c. Show that for every $n \in \mathbb{N}$, $I_n = \{a \in F \mid \nu(a) \geq n\}$ is a proper ideal of R. I_1 is called the *valuation ideal*.

Exercise 4. Let F be a field with discrete valuation ν, valuation ring R, and ideals I_n as in the previous exercise. A *uniformizer* of R is an element $\pi \in F$ with $\nu(\pi) = 1$.

a. Let π be a uniformizer of R. Prove that $I_n = \pi^n R$.

b. Prove that $\{0\} \subsetneq \cdots \subsetneq I_3 \subsetneq I_2 \subsetneq I_1 \subsetneq R$.

c. Show that $\bigcap_{n \in \mathbb{N}} I_n = \{0\}$.

d. Let $A \subset R$ be a nonzero proper ideal. Show that there is an $n \in \mathbb{N}$ so that $A = I_n$.