Modern Algebra II
Fall 2019

Assignment 3.3
Due September 18

Exercise 1. Let $I = (X)$ be the principal ideal generated by X in $\mathbb{Z}[X]$. Prove that for any $n \geq 2$ there is a strictly increasing chain

$$I = I_1 \subsetneqq I_2 \subsetneqq \cdots \subsetneqq I_n$$

of ideals in $\mathbb{Z}[X]$ (*cf.* exercise 3.1.3).

Exercise 2. Show that multiplication of ideals is associative.

Exercise 3. Let A and B be ideals in a ring R. For $n \in \mathbb{N}$ let

$$nA = \underbrace{A + A + \cdots + A}_{n \text{ summands}},$$
$$A^n = \underbrace{AA \cdots A}_{n \text{ factors}}.$$

a. Prove that $nA = A$ for all $n \geq 1$.

b. Show that $\cdots \subset A^3 \subset A^2 \subset A$.

c. If $A + B = R$, prove that $A^n + B^n = R$ for all $n \geq 1$.

Exercise 4. Let R be a commutative principal ideal ring in which $A^2 = A$ for every ideal A. Prove that every nonzero element of R is either a unit or a zero divisor (compare to exercise 1.2.4). With a good deal more work one can show that, in fact, any such ring is isomorphic to a product of finitely many fields.