Modern Algebra II
Fall 2019

Assignment 4.1
Due October 2

Exercise 1. Let $\varphi : R \to R_1$ and $\psi : R \to R_2$ be homomorphisms of rings.

a. Check that the product map $\varphi \times \psi : R \to R_1 \times R_2$ is a ring homomorphism.

b. Prove that $\varphi \times \psi : R \to R_1 \times R_2$ is a surjection if and only if (i) φ and ψ are surjections and (ii) $(1_{R_1}, 0), (0, 1_{R_2}) \in \operatorname{im} \varphi \times \psi$.

Exercise 2. Let R be a ring and $\mathcal{I}$ a collection of ideals in R. We say that the ideals in $\mathcal{I}$ are *pairwise coprime* if $A + B = R$ for all distinct $A, B \in \mathcal{I}$.

a. Prove that if $A_1, A_2, \ldots, A_m$ are pairwise coprime ideals in a commutative ring R, then $A_1 A_2 \cdots A_{m-1} + A_m = R$.

b. Prove that if $A_1, A_2, \ldots, A_n$ are pairwise coprime ideals in a commutative ring R, then

$$A_1 \cap A_2 \cap \cdots \cap A_n = A_1 A_2 \cdots A_n.$$

Exercise 3. Let A, B be a ideals in a ring R. Suppose that $A \subset B$ and let

$$C = \{r \in R \,|\, br \in A \text{ for all } b \in B\}.$$

Prove that C is an ideal in R containing A, and that

$$BC \subset A \subset B \cap C.$$

Exercise 4. Let R_1 and R_2 be rings. For $k = 1, 2$, let $\pi_k : R_1 \times R_2 \to R_k$ denote projection onto the kth coordinate.

a. Prove that π_k is a ring homomorphism for $k = 1, 2$.

b. Let $I \subset R_1 \times R_2$. Prove that I is an ideal in $R_1 \times R_2$ if and only if $I = A_1 \times A_2$ for some ideals $A_k \subset R_k$, $k = 1, 2$. [*Suggestion:* Apply exercise 3.2.1 to the projection maps.]

c. Prove that $(R_1 \times R_2)^\times = R_1^\times \times R_2^\times$.