Exercise 1. Prove that a commutative ring R is a domain if and only if (0) is a prime ideal.

Exercise 2. A commutative ring R is called *local* if it has a unique maximal ideal. Prove that a commutative ring R is a local ring with maximal ideal M if and only if $M = R \setminus R^\times$.

Exercise 3. The *spectrum* $\operatorname{Spec} R$ of a commutative ring R is defined to be the set of all prime ideals in R. Given an ideal $A \subset R$, the *ideal variety* in $\operatorname{Spec} R$ associated to A is

$$V(A) = \{P \in \operatorname{Spec} R \mid A \subset P\}.$$

Let $B \subset R$ be an ideal.

a. Prove that $A \subset B$ implies $V(B) \subset V(A)$.

b. Prove that $V(A + B) = V(A) \cap V(B)$.

c. Prove that $V(AB) = V(A) \cup V(B)$.

Remark. Part **b** can be generalized as follows. If $\mathcal{I}$ is any collection of ideals, and we define the sum of the ideals in $\mathcal{I}$ by

$$\sum_{A\in\mathcal{I}} A = \left(\bigcup_{A\in\mathcal{I}} A\right),$$

we leave it as an additional exercise to show that $V(\sum_{A\in\mathcal{I}} A) = \bigcap_{A\in\mathcal{I}} V(A)$. Together with part **c** this proves that the ideal varieties $V(I)$ are the closed sets of a topology on $\operatorname{Spec} R$, known as the *Zariski topology.* In modern algebraic geometry, the space $\operatorname{Spec} R$ equipped with the Zariski topology (and some additional algebraic information) is called an *affine scheme.*

Exercise 4. Let I be an interval of real numbers and $R = C^0(I)$ be the set of all continuous real-valued functions on I. Note that R is a subring of $\mathbb{R}^I$ by standard results from calculus. Prove that for each $a \in I$, the set

$$M_a = \{f \in R \mid f(a) = 0\}$$

is a maximal ideal in R. Are these the only maximal ideals in R?