Modern Algebra II
Fall 2019

Assignment 6.1
Due October 23

Exercise 1. Let F be a finite field with q elements. Let $E : F[X] \to F^F$ be the map that sends each polynomial to the function it defines on F. That is, $E(f)(a) = f(a)$ for all $a \in F$.

- **a.** Show that every element of F is a root of $\phi(X) = X^q - X \in F[X]$.
- **b.** Show that if $f(X) \in F[X]$ satisfies $f(a) = 0$ for all $a \in F$, then $f(X) = g(X)\phi(X)$ for some $g(X) \in F[X]$.
- **c.** Prove that E is a surjective homomorphism with kernel $(\phi(X))$, so that

$$F[X]/(\phi(X)) \cong F^F.$$

[*Suggestions:* Use the fact that the pointwise evaluation maps E_a are homomorphisms. Surjectivity can be established directly, but it may be easier to use the pigeonhole principle.]

Exercise 2. Let R be a Euclidean domain with norm N.

- **a.** Let $a \in R$. Prove that if $a \neq 0$ and $N(a) = 0$, then $a \in R^\times$.
- **b.** Prove that $N(1) \leq N(a)$ for all nonzero $a \in R$.
- **c.** Let $a \in R$, $a \neq 0$. Show that $N(a) = N(1)$ if and only if $a \in R^\times$.
- **d.** Determine $\mathbb{Z}[i]^\times$.

Exercise 3. Find the quotient and remainder when $7 + 10i$ is divided by $2 - 3i$ in $\mathbb{Z}[i]$. Show that there are four possible quotient-remainder pairs when 17 is divided by $3 - 5i$.

Exercise 4. Let $\alpha = \frac{1+\sqrt{-3}}{2} \in \mathbb{C}$.

- **a.** Show that α is a root of a monic quadratic polynomial in $\mathbb{Z}[X]$.
- **b.** Prove that for every $z \in \mathbb{C}$ there is a $w \in \mathcal{P} = \{s + t\alpha \mid s, t \in [0, 1)\}$ so that $z - w \in \mathbb{Z}[\alpha]$. That is, $\mathbb{C} = \mathcal{P} + \mathbb{Z}[\alpha]$. [*Suggestion:* Start by showing that $\{1, \alpha\}$ is an $\mathbb{R}$-basis for $\mathbb{C}$.]
- **c.** Exploit the symmetry of $\mathcal{P}$ to prove that $\mathbb{Z}[\alpha]$ with the norm $N(z) = |z|^2$ is a Euclidean domain.
- **d.** Show that $\mathbb{Z}[\alpha]^\times = \{\pm 1, \pm\alpha, \pm\overline{\alpha}\}$.