Modern Algebra II
Fall 2019

Assignment 6.2
Due October 23

Exercise 1. Let R be a commutative ring and let $A, B, C \subset R$ be ideals.

- **a.** If $A + B = R$, prove that $A^m + B^n = R$ for all $m, n \in \mathbb{N}$. [*Suggestion:* This follows easily from Exercise 3.3.3, but can also be proven directly.]
- **b.** If A and B are coprime and $AB = C^n$ for some $n \in \mathbb{N}$, prove that there are coprime ideals A_0 and B_0 so that $A = A_0^n$, $B = B_0^n$ and $A_0B_0 = C$. [*Suggestion:* Take $A_0 = A+C$ and $B_0 = B + C$.]

Exercise 2. Let R be a PID, $a, b, c \in R$ and $n \in \mathbb{N}$. If a and b are coprime[1] and $ab = c^n$, prove that there exist coprime $a_0, b_0 \in R$ and $u, v, w \in R^\times$ so that $a = ua_0^n$, $b = vb_0^n$ and $a_0b_0 = wc$.

Exercise 3. Let $a, b \in \mathbb{Z}$. Prove that if b is even and $\gcd(a, b) = 1$, then $(a+bi, a-bi) = \mathbb{Z}[i]$.

Exercise 4. A *Pythagorean triple* is a 3-tuple $(a, b, c) \in \mathbb{Z}^3$ so that $a^2 + b^2 = c^2$. A Pythagorean triple is called *primitive* if $\gcd(a, b, c) = 1$.

- **a.** If (a, b, c) is a primitive Pythagorean triple, prove that a and b have opposite parity. [*Suggestion:* Work modulo 4 and argue by contradiction.]
- **b.** Show that if (a, b, c) is a primitive Pythagorean triple and b is even, then there exist $m, n \in \mathbb{Z}$ so that $\gcd(m, n) = 1$ and
$$\begin{aligned} a &= m^2 - n^2, \\ b &= 2mn, \\ c &= \pm(m^2 + n^2). \end{aligned}$$
- **c.** Prove that m and n in part **b** must have opposite parity. Conclude that the even member of a primitive pythagorean triple is always divisible by 4.

[1] We say $a, b \in R$ are coprime provided (a) and (b) are coprime ideals.