Modern Algebra II
Fall 2019

Assignment 6.3
Due October 23

Exercise 1. Suppose $\alpha \in \mathbb{C}$ is a nonreal root of $X^2+pX+q \in \mathbb{Z}[X]$, so that $\Delta = p^2-4q < 0$. In class we showed that for $z = a + b\alpha \in Z[\alpha]$, one has $N(z) = |z|^2 = a^2 - pab + qb^2 \in \mathbb{N}_0$.

a. By considering Δ modulo 4, show that $\Delta \leq -3$.

b. When $\Delta = -3$, show that $\mathbb{Z}[\alpha] = \mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right]$. When $\Delta = -4$, show that $\mathbb{Z}[\alpha] = \mathbb{Z}[i]$. The unit groups of these quadratic orders were computed in Assignment 6.1.

c. Show that $z \in \mathbb{Z}[\alpha]^\times$ if and only if $N(z) = 1$.

d. Show that $N(a + b\alpha) = 1$ if and only if $(2a - bp)^2 + |\Delta|b^2 = 4$.

e. When $\Delta \leq -5$, show that $\mathbb{Z}[\alpha]^\times = \{\pm 1\}$.

So, with only two exceptions, the unit group in $\mathbb{Z}[\alpha]$ contains only ± 1.

Exercise 2. Let $\mathbb{Z}[\alpha]$ and $N(\cdot)$ be as above. Show that for any $z \in \mathbb{Z}[\alpha]$, if $N(z)$ is a prime integer, then z is irreducible.

Exercise 3. Use the norm and Exercise 1 to prove that 3 and $2 \pm \sqrt{-5}$ are nonassociate irreducibles in $\mathbb{Z}[\sqrt{-5}]$.

Exercise 4. Let D be a domain. Prove that if $a_0 \in D$ is irreducible, then $f = \sum_{i \geq 0} a_i X^i \in D[[X]]$ is irreducible. Show that this statement is false if $D[[X]]$ is replaced by $D[X]$.