Exercise 1. Let S be a nonempty subset of a commutative ring R. An element $d \in R$ is a *greatest common divisor* (GCD) of S provided:

(i) $d|a$ for all $a \in S$;

(ii) $c \in R$ and $c|a$ for all $a \in S$ implies $c|d$.

Prove the following.

a. S has GCD d if and only if $\{(c) \,|\, (S) \subset (c)\}$ has least element (d) (under inclusion).

b. Any two GCDs of S are associate.

Exercise 2. A *Bézout domain* is a domain in which every finitely generated ideal is principal. A *GCD domain* is a domain with the property that every nonempty finite subset has a GCD. Prove the following.

a. Every PID is a GCD domain.

b. Every Bézout domain is a GCD domain.

c. Every Noetherian Bézout domain is a PID.

d. Every UFD is a GCD domain.

[*Suggestions:* Use Exercise **1a** for parts **a** - **c**. For part **d**, given finitely many elements in a UFD, use their prime factorizations to construct a GCD.]

Exercise 3. Prove that the image or inverse image of a multiplicative set under a ring homomorphism is also a multiplicative set.

Exercise 4. Prove that the intersection of any collection of multiplicative sets in a ring R is again a multiplicative set. In particular, the complement of a union of prime ideals in R is a multiplicative set.