Modern Algebra II
Fall 2019

Assignment 8.1
Due November 6

Exercise 1. Let R be a commutative ring, $S \subset R$ be a multiplicative set, and $\varphi_S : R \to S^{-1}R$ the canonical map $a \mapsto \frac{a}{1}$. Prove that φ_S is an embedding if and only if S is proper.

Exercise 2. Let R, S and φ_S be as in the preceding exercise. Let $\psi : R \to R'$ be a ring homomorphism so that $\psi(S) \subset (R')^\times$, and let $\widehat{\psi} : S^{-1}R \to R'$ be the induced map given by $\widehat{\psi}\left(\frac{a}{s}\right) = \psi(a)\psi(s)^{-1}$. Prove that when S is proper, $\widehat{\psi}$ is injective if and only if ψ is injective.

Exercise 3. Let p be a prime, $\pi : \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ be the canonical projection and $S = \mathbb{Z} \dashv p\mathbb{Z}$.

a. Prove that $\pi(S) \subset (\mathbb{Z}/p\mathbb{Z})^\times$.

b. Let $\widehat{\pi} : \mathbb{Z}_{(p)} \to \mathbb{Z}/p\mathbb{Z}$ denote the induced map. Show that

$$\ker \widehat{\pi} = p\mathbb{Z}_{(p)} = \left\{ \frac{a}{s} \,\middle|\, a, s \in \mathbb{Z},\ p|a,\ p \nmid s \right\}.$$

c. Conclude that $\mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)} \cong \mathbb{Z}/p\mathbb{Z}$, and hence that $p\mathbb{Z}_{(p)}$ is a maximal ideal in $\mathbb{Z}_{(p)}$.