Exercise 1. Let R be a commutative ring and $S \subset R^\times$ a multiplicative set. Show that the identity map $I : R \to R$ lifts to an isomorphism $\widehat{I} : S^{-1}R \xrightarrow{\sim} R$. That is, localizing with elements that are already units in R has no effect on R. [*Suggestion:* Show that $\widehat{I}^{-1} = \varphi_S$.]

Exercise 2. Let R be a commutative ring, $A \subset R$ an ideal, and $S \subset R$ a multiplicative set. Let $\pi : R \to R/A$ be the canonical surjection and $S/A = \pi(S) = \{s + A \mid s \in S\}$. Recall that since π is a homomorphism, S/A is a multiplicative subset of R/A.

- **a.** Show that $\psi = \varphi_{S/A} \circ \pi : R \to (S/A)^{-1}(R/A)$ maps S into the unit group of $(S/A)^{-1}(R/A)$.
- **b.** Let $\widehat{\psi} : S^{-1}R \to (S/A)^{-1}(R/A)$ denote the induced map. Prove that $\widehat{\psi}$ is surjective and $\ker \widehat{\psi} = S^{-1}A$. Conclude that $(S^{-1}R)/(S^{-1}A) \cong (S/A)^{-1}(R/A)$.

This shows that the operations "localize at S" and "divide by A" commute.

Exercise 3. Let R be a commutative ring and S a multiplicative set. Let $P \subset R$ be a prime ideal such that $P \cap S = \varnothing$.

- **a.** Prove that $P = \varphi_S^{-1}(S^{-1}P)$.
- **b.** Prove that $S^{-1}P$ is a prime ideal in $S^{-1}R$. [*Remark:* Don't forget to prove it's proper!]