On the Complex Matrix Representation of the Quaternions

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1 Introduction

Although it can be constructed rigorously in a number of ways, the division ring $\mathbb{H}$ of Hamiltonian quaternions is classically defined to be the set of all formal linear combinations of the form

$$\alpha = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k},$$

where $a, b, c, d \in \mathbb{R}$. Elements of $\mathbb{H}$ are added coordinate-wise, and multiplied formally, allowing scalars to commute with other elements, using the relationships

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1.$$

The center of $\mathbb{H}$ is the set of *real quaternions*, those quaternions which satisfy $b = c = d = 0$, and it is clearly isomorphic to $\mathbb{R}$. The quaternions of the form $a + b\mathbf{i}$ (i.e. $c = d = 0$) also form a subring of $\mathbb{H}$, this time isomorphic to $\mathbb{C}$.[1]

Note that the real quaternions embed into the "complex quaternions" in the usual way, namely as those elements with $b = 0$. It follows that we can view both $\mathbb{R}$ and $\mathbb{C}$ as subrings of $\mathbb{H}$, with the usual inclusion $\mathbb{R} \subset \mathbb{C}$. While $\mathbb{R}$ is central in $\mathbb{H}$, $\mathbb{C}$ is not. However, because the units $\mathbf{i}, \mathbf{j}, \mathbf{k}$ anti-commute, for $z = a + b\mathbf{i} \in \mathbb{C}$ we have the commutation relations

$$\begin{aligned} z\mathbf{j} &= a\mathbf{j} + b\mathbf{k} = \mathbf{j}(a - b\mathbf{i}) = \mathbf{j}\overline{z}, \\ z\mathbf{k} &= a\mathbf{k} - b\mathbf{j} = \mathbf{k}(a - b\mathbf{i}) = \mathbf{k}\overline{z}. \end{aligned} \tag{1}$$

The goal of this note is to explicitly realize $\mathbb{H}$ as a ring of 2×2 matrices over $\mathbb{C}$. More specifically, we will prove that

$$\mathbb{H} \cong \left\{ \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix} \,\middle|\, z, w \in \mathbb{C} \right\}.$$

The conjugations on the right-hand side are a result of the commutation relations (1). Without them the ring on the right would be commutative and contain zero divisors.

2 The Regular Representation

Let R be a ring containing a field F in its center. For example, $R = \mathrm{M}_n(F)$ with F any field, embedded as the scalar matrices αI, $\alpha \in F$. Left multiplication by F turns R into an F-vector space, and it isn't hard to show that $\mathrm{End}_F(R)$, the set of F-linear endomorphisms of R, is a subring of the full endomorphism ring $\mathrm{End}\, R$.[2] There is a natural way to embed R into $\mathrm{End}_F(R)$ by viewing left multiplication in R as an F-linear operation. That is, for each $a \in R$, define $\lambda_a : R \to R$ by $\lambda_a(x) = ax$. For any $x, y \in R$ and $\alpha \in F$ we then have

$$\lambda_a(x + y) = a(x + y) = ax + ay = \lambda_a(x) + \lambda_a(y),$$

[1] It's tempting to call this the subring of *complex quaternions*, but in the literature the term *complex* is used to refer to any quaternion having at least one of b, c, d nonzero.

[2] Recall that here we are dealing with the endomorphisms of $(R, +)$.

and since F is central in R, for any $\alpha \in F$,

$$\lambda_a(\alpha x) = a(\alpha x) = (a\alpha)x = (\alpha a)x = \alpha(ax) = \alpha\lambda_a(x).$$

That is, $\lambda_a \in \mathrm{End}_F(R)$. Since $\lambda_{a+b} = \lambda_a + \lambda_b$, $\lambda_{ab} = \lambda_a \circ \lambda_b$ and $\lambda_{1_R} = \mathrm{id}$ (easy exercises), we find that the map

$$\begin{aligned} \rho : R &\to \mathrm{End}_F(R) \\ a &\mapsto \lambda_a \end{aligned}$$

is a homomorphism of rings. From $\lambda_a(1_R) = a1_R = a$ it follows that $\lambda_a \equiv 0$ if and only if $a = 0$. Hence the kernel of ρ is trivial, and we conclude that ρ is an embedding. ρ is called the *left regular representation* of R.

When $n = \dim_F R$ is finite, we can make ρ a bit more concrete. Given an F-basis $\mathcal{B}$ for R, it is well-known that the coordinate map

$$\begin{aligned} [\,\cdot\,]_{\mathcal{B}} : \mathrm{End}_F(R) &\to \mathrm{M}_n(F) \\ T &\longmapsto [T]_{\mathcal{B}}, \end{aligned}$$

where $[T]_{\mathcal{B}}$ denotes the matrix of T relative to $\mathcal{B}$, is an isomorphism. It follows that the composite map, $\rho_{\mathcal{B}}(a) = [\lambda_a]_{\mathcal{B}}$, is an embedding of R into the matrix ring $\mathrm{M}_n(F)$. When the field F is familiar, this can be a convenient way to realize more abstract rings that contain F centrally.

When F is not central in R, the construction of the left regular representation fails: λ_a may not preserve scalar multiplication. The problem is that F and R are both acting on the left. The easiest way out of this situation is to simply let R act on the right instead. For $a, x \in R$, define $\mu_a(x) = xa$. As above, μ_a is additive, and for $\alpha \in F$ we have

$$\mu_a(\alpha x) = (\alpha x)a = \alpha(xa) = \alpha\mu_a(x),$$

so that μ_a does preserve scalar multiplication of R by F. Thus $\mu_a \in \mathrm{End}_F(R)$. However, as one can easily check, for $a, b \in R$ we have $\mu_{ab} = \mu_b \circ \mu_a$, so that the rule $a \mapsto \mu_a$ need not define a homomorphism from R to $\mathrm{End}_F(R)$.

When R has finite dimension n over F, we can remedy this situation with the transpose. As above, let $\mathcal{B}$ be an F-basis for R, and consider $\sigma_{\mathcal{B}} : R \to \mathrm{End}_F(R)$ given by $a \mapsto [\mu_a]^t_{\mathcal{B}}$. Because the transpose is F-linear and reverses the order of multiplication, we immediately conclude that $\sigma_{\mathcal{B}}$ is a homomorphism. As above, composition with evaluation at 1_R proves that $\sigma_{\mathcal{B}}$ is injective. We call $\sigma_{\mathcal{B}}$ the *right regular representation* of R.[3]

In the case that R is infinite dimensional (e.g. $R = \mathbb{R}$ and $F = \mathbb{Q}$), $\mathrm{End}_F(R)$ is no longer isomorphic to a matrix ring, and "correcting" μ becomes a bit more complicated. The transpose of a linear endomorphism can be defined abstractly, but it requires the notion of a *dual vector space*, and at the end of the day one ends up with an embedding of R into the endomorphism ring of its (vector space) dual, instead of $\mathrm{End}_F(R)$.

3 The Quaternions as a Complex Matrix Algebra

As a vector space over $\mathbb{R}$, the ring $\mathbb{H}$ is four-dimensional, by definition. Therefore, the left regular representation of $\mathbb{H}$ over $\mathbb{R}$ will yield a subring of $\mathrm{M}_4(\mathbb{R})$ isomorphic to $\mathbb{H}$. To compute it, first set $q = a + b\mathbf{i} + c\mathbf{j} + d\,k$. We then have

$$\begin{aligned} \lambda_q(1) &= q \cdot 1 = q, \\ \lambda_q(\mathbf{i}) &= q \cdot \mathbf{i} = -b + a\mathbf{i} + d\mathbf{j} - c\mathbf{k}, \\ \lambda_q(\mathbf{j}) &= q \cdot \mathbf{j} = -c - d\mathbf{i} + a\mathbf{j} + b\mathbf{k}, \\ \lambda_q(\mathbf{k}) &= q \cdot \mathbf{k} = -d + c\mathbf{i} - b\mathbf{j} + a\mathbf{k}. \end{aligned}$$

[3]Strictly speaking, using the definite article "the" is inappropriate, because $\sigma_{\mathcal{B}}$ depends on the choice of $\mathcal{B}$. However, we will overlook this slight abuse of terminology.

So with $\mathcal{B} = \{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$, we find that

$$\rho_{\mathcal{B}}(q) = [\lambda_q]_{\mathcal{B}} = \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix}. \tag{2}$$

It follows at once that $\mathbb{H}$ is isomorphic to the ring of all real matrices of the form (2).

Since

$$a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} = \underbrace{a + b\mathbf{i}}_{z \in \mathbb{C}} + (\underbrace{c + d\mathbf{i}}_{w \in \mathbb{C}})\mathbf{j} = z + w\mathbf{j},$$

the set $\mathcal{B} = \{1, \mathbf{j}\}$ is a $\mathbb{C}$-basis for $\mathbb{H}$, so that the dimension of $\mathbb{H}$ over $\mathbb{C}$ is 2. Because $\mathbb{C}$ is not central in $\mathbb{H}$, only the right regular representation over $\mathbb{C}$ is defined. For $z, w \in \mathbb{C}$ we have

$$\mu_{z+w\mathbf{j}}(1) = 1(z + w\mathbf{j}) = z + w\mathbf{j},$$
$$\mu_{z+w\mathbf{j}}(\mathbf{j}) = \mathbf{j}(z + w\mathbf{j}) = \overline{z}\mathbf{j} + \overline{w}\mathbf{j}^2 = -\overline{w} + \overline{z}\mathbf{j}.$$

Consequently

$$[\mu_{z+w\mathbf{j}}]_{\mathcal{B}} = \begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix},$$

so that

$$\sigma_{\mathcal{B}}(z + w\mathbf{j}) = [\mu_{z+w\mathbf{j}}]_{\mathcal{B}}^t = \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix}.$$

Hence the image of $\sigma_{\mathcal{B}}$, namely

$$\mathbf{H} = \left\{ \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix} \,\middle|\, z, w \in \mathbb{C} \right\},$$

is a subring of $\mathrm{M}_2(\mathbb{C})$ that is isomorphic to $\mathbb{H}$, which is precisely what we sought to prove.