Linear Approximations and Differentiability

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Calculus III

Introduction

The notion of differentiability is central to the study of calculus.

In one variable, the differentiability of $f(x)$ amounts to the existence of the derivative $f'(x)$.

In multiple variables, however, the existence of partial derivatives *is not* equivalent to differentiability.

To be truly differentiable a function must be “locally linear,” a feature that partial derivatives alone cannot capture.

Recall

A function $f(x)$ is *differentiable* at $x = a$ provided

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad \text{exists.}$$

In this case the *linear approximation to f at a* is

$$L(x) = f'(a)(x - a) + f(a),$$

which is just the function whose graph is the tangent line.

Question: How should we define differentiability for functions of two or more variables?

First Attempt

Given $f(x,y)$ and a point (a,b), we might say that f is differentiable at (a,b) provided

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y)-f(a,b)}{\sqrt{(x-a)^2+(y-b)^2}}$$

exists.

However, this limit usually fails to exist!

For instance, if we approach (a,b) parallel to the coordinate axes we get the partial derivatives $f_x(a,b)$ and $f_y(a,b)$, which need not agree.

Reinterpretation

To find out what differentiability "really" means, let's return to the single variable case:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

This is equivalent to

$$\lim_{x \to a} \left(\frac{f(x) - f(a)}{x - a} - f'(a) \right) = 0.$$

But

$$\frac{f(x) - f(a)}{x - a} - f'(a) = \frac{f(x) - f(a) - f'(a)(x - a)}{x - a} = \frac{f(x) - L(x)}{x - a}.$$

That is, f is differentiable at $= a$ provided

$$\lim_{x \to a} \frac{f(x) - L(x)}{x - a} = 0,$$

where L is the linear approximation to f at a.

This can be interpreted as saying that as $x \to a$, L becomes a "very good" approximation to f.

Put another way, as we zoom in on the point $(a, f(a))$, the graph of f becomes linear.

It turns out that *this* idea carries over to multiple variables very easily.

Second Attempt

Definition

Given a function $f(x, y)$ and a point (a, b), the *linear approximation* to f at (a, b) is

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

The graph of L is a plane. Since

$$L(a, b) = f(a, b) \quad \text{and} \quad L_x(a, b) = f_x(a, b) \quad \text{and} \quad L_y(a, b) = f_y(a, b),$$

the graph of f is tangent to this plane *in the x and y directions.* We will define f to be *differentiable* at (a, b) provided L is a "good approximation" to f as $(x, y) \to (a, b)$, akin to the one variable situation.

Definition

Given a function $f(x, y)$ and a point (a, b), we say that f is *differentiable* at (a, b) provided

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - L(x,y)}{\sqrt{(x-a)^2 + (y-b)^2}} = 0,$$

where L is the linear approximation to f at (a, b).

Remarks.

- The graph of a differentiable function becomes planar as we zoom in on any point.
- We can define linear approximations and differentiability in any number of variables in a similar way.

It is useful to have criteria for differentiability that don't involve the evaluation of a multivariate limit.

Fortunately there is one that is particularly simple.

Theorem 1

If f_x and f_y are continuous at (a, b), then f is differentiable at (a, b).

Example. Let $f(x, y) = 4x^2 - y^2 + 2y$. Then $f_x(x, y) = 8x$ and $f_y(x, y) = -2y + 2$.

Since f_x and f_y are polynomials, they are continuous everywhere.

It follows that f is differentiable everywhere.

Example 1

Let $f(x, y) = \sqrt{20 - x^2 - 7y^2}$. Use a linear approximation to estimate $f(1.99, 1.01)$.

Solution. Since

$$f_x(x, y) = \frac{-x}{\sqrt{20 - x^2 - 7y^2}} \quad \text{and} \quad f_y(x, y) = \frac{-7y}{\sqrt{20 - x^2 - 7y^2}}$$

are continuous where $20 - x^2 - 7y^2 > 0$, f is differentiable at the point $(2, 1)$.

Thus $f(1.99, 1.01) \approx L(1.99, 1.01)$, where L is the linear approximation to f at $(2, 1)$.

Because

$$f(2,1) = \sqrt{20-4-7} = \sqrt{9} = 3,$$

$$f_x(2,1) = \frac{-2}{\sqrt{20-4-7}} = \frac{-2}{3},$$

$$f_y(2,1) = \frac{-7}{\sqrt{20-4-7}} = \frac{-7}{3},$$

the linear approximation is

$$L(x,y) = 3 - \frac{2}{3}(x-2) - \frac{7}{3}(y-1).$$

Therefore

$$f(1.99, 1.01) \approx L(1.99, 1.01) = 3 - \frac{2}{3}(-0.01) - \frac{7}{3}(0.01)$$

$$= \boxed{\frac{895}{300} = 2.98333\ldots}$$

Differentials

Definition

The *differential* of $f(x, y)$ is $df = f_x(x, y)\,dx + f_y(x, y)\,dy$.

If we change (a, b) to $(a + dx, b + dy)$, it is not hard to see that

$$df = \text{ exact change in } L.$$

It follows that if f is differentiable at (a, b), then

$$df \approx \text{ change in } f.$$

Remark. We have an analogous definition and interpretation in any number of variables.

Example

The differential of $f(x,y,z) = e^{x^2+yz}$ is

$$df = 2xe^{x^2+yz}\,dx + ze^{x^2+yx}\,dy + ye^{x^2+yz}\,dz.$$

Because the partial derivatives are continuous everywhere, f is differentiable everywhere.

So, if we move from $(0,2,0)$ to $(0.1, 1.95, 0.01)$, the approximate change in f is

$$df = 2 \cdot 0 \cdot 0.1 + 0 \cdot (-0.05) + 2e^{0^2+0} \cdot 0.01 = 0.02.$$

Another Example

Example 2

Four positive numbers are rounded to the first decimal place and multiplied together. Estimate the absolute relative error in the product introduced by rounding.

Solution. Denote the four numbers by n_1, n_2, n_3 and n_4, and their product by P:

$$P = n_1 n_2 n_3 n_4.$$

The differential of P is

$$\begin{aligned} dP &= n_2 n_3 n_4 \, dn_1 + n_1 n_3 n_4 \, dn_2 + n_1 n_2 n_4 \, dn_3 + n_1 n_2 n_3 \, dn_4 \\ &= \frac{P}{n_1} \, dn_1 + \frac{P}{n_2} \, dn_2 + \frac{P}{n_3} \, dn_3 + \frac{P}{n_4} \, dn_4. \end{aligned}$$

The approximate relative error introduced by changing n_1, n_2, n_3 and n_4 is therefore

$$\frac{dP}{P} = \frac{dn_1}{n_1} + \frac{dn_2}{n_2} + \frac{dn_3}{n_3} + \frac{dn_4}{n_4}.$$

When rounding, one has $|dn_i| \leq 0.05$ for all i.

Thus

$$\left|\frac{dP}{P}\right| \leq \boxed{0.05\left(\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} + \frac{1}{n_4}\right).}$$