Directional Derivatives and the Gradient Vector

Ryan C. Daileda

Trinity University

Calculus III

Introduction

Directional Derivatives

Given a function $f(x, y)$, a point (x_0, y_0) and vector $\mathbf{v} = \langle a, b \rangle$ we ask: what is the "slope" of the graph of f at (x_0, y_0) *in the* **v***-direction*?

The partial derivatives f_x and f_y answer this question when $\mathbf{v} = \mathbf{i}, \mathbf{j}$.

As with f_x and f_y, we can express the answer as a limit of difference quotients.

The line through (x_0, y_0) in the **v**-direction has the parametric form

$$x = x_0 + at,$$
$$y = y_0 + bt.$$

The slope between $(x_0, y_0, f(x_0, y_0))$ and $(x, y, f(x, y))$ is

$$\frac{f(x,y) - f(x_0,y_0)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = \frac{f(x_0+at, y_0+bt) - f(x_0,y_0)}{\sqrt{((x_0+at)-x_0)^2 + ((y_0+bt)-y_0)^2}}$$

$$= \frac{f(x_0+at, y_0+bt) - f(x_0,y_0)}{\sqrt{a^2t^2 + b^2t^2}}$$

$$= \frac{f(x_0+at, y_0+bt) - f(x_0,y_0)}{t|\mathbf{v}|}.$$

Definition

We define the *directional derivative* of $f(x, y)$ at (x_0, y_0) in the $\mathbf{v} = \langle a, b \rangle$-direction to be

$$D_{\mathbf{v}} f(x_0, y_0) = \lim_{t \to 0} \frac{f(x_0+at, y_0+bt) - f(x_0,y_0)}{t|\mathbf{v}|}.$$

Computing the Directional Derivative

To compute $D_{\mathbf{v}}f$, we introduce the linear approximation at (x_0, y_0):

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0).$$

We then examine the effect of "replacing" f by L in the limit defining $D_{\mathbf{v}}f$:

$$\frac{f(x_0+at, y_0+bt) - f(x_0,y_0)}{t|\mathbf{v}|} = \\ = \frac{f(x_0+at, y_0+bt) - L(x_0+at, y_0+bt)}{t|\mathbf{v}|} \\ + \frac{L(x_0+at, y_0+bt) - f(x_0,y_0)}{t|\mathbf{v}|}$$

$$= \frac{f(x_0 + at, y_0 + bt) - L(x_0 + at, y_0 + bt)}{t|\mathbf{v}|} + \frac{f_x(x_0, y_0)at + f_y(x_0, y_0)bt}{t|\mathbf{v}|}$$

$$= \frac{f(x_0 + at, y_0 + bt) - L(x_0 + at, y_0 + bt)}{t|\mathbf{v}|} + \frac{f_x(x_0, y_0)a + f_y(x_0, y_0)b}{|\mathbf{v}|}.$$

If f is differentiable at (x_0, y_0), then

$$\lim_{t \to 0} \frac{f(x_0 + at, y_0 + bt) - L(x_0 + at, y_0 + bt)}{t|\mathbf{v}|} = 0.$$

We conclude that:

Theorem 1

If $f(x, y)$ is differentiable at (x_0, y_0) and $\mathbf{v} = \langle a, b \rangle$, then

$$D_{\mathbf{v}} f(x_0, y_0) = \frac{f_x(x_0, y_0)a + f_y(x_0, y_0)b}{|\mathbf{v}|}.$$

Remarks.

- Notice that when $\mathbf{v} = \mathbf{i}, \mathbf{j}$ we recover f_x and f_y.
- The textbook assumes that $\mathbf{v}$ is a unit vector, and therefore does not divide by $|\mathbf{v}|$ in its formula for $D_{\mathbf{v}} f$.

Example 1

Find the directional derivative of $f(x, y) = 1 + 2x\sqrt{y}$ at $(3, 4)$ in the direction of $\mathbf{v} = \langle 4, -3 \rangle$.

Solution. First we compute the partial derivatives:

$$f_x(x, y) = 2\sqrt{y}, \quad f_y(x, y) = \frac{x}{\sqrt{y}}.$$

Thus

$$f_x(3, 4) = 2\sqrt{4} = 4, \quad f_y(3, 4) = \frac{3}{\sqrt{4}} = \frac{3}{2}.$$

Therefore

$$D_{\mathbf{v}} f(3, 4) = \frac{4 \cdot 4 + \frac{3}{2} \cdot (-3)}{\sqrt{4^2 + (-3)^2}} = \frac{16 - \frac{9}{2}}{5} = \frac{23}{10} = \boxed{2.3}$$

The Gradient Vector

Definition

The *gradient (vector)* of $f(x, y)$ is

$$\nabla f = \langle f_x(x, y), f_y(x, y) \rangle.$$

Examples.

1. The gradient of $f(x, y) = 1 + 2x\sqrt{y}$ is
$$\nabla f = \left\langle 2\sqrt{y}, \frac{x}{\sqrt{y}} \right\rangle.$$
2. The gradient of $f(x, y) = 2x^2 + 3y^2 - 5xy$ is
$$\nabla f = \langle 4x - 5y, 6y - 5x \rangle.$$

Remarks

- The gradient is our first example of a *vector field*.
- When f is differentiable, we regard ∇f as its *derivative*.
- Notice that if $\mathbf{v} = \langle a, b \rangle$, then

$$D_{\mathbf{v}} f = \frac{f_x a + f_y b}{|\mathbf{v}|} = \frac{\langle f_x, f_y \rangle \cdot \langle a, b \rangle}{|\mathbf{v}|} = \boxed{\frac{\nabla f \cdot \mathbf{v}}{|\mathbf{v}|}}.$$

Example 2

Find the directional derivative of $f(x,y) = 2x^2 - 3y^2 + 5xy$ at $(1,1)$ toward the origin.

Solution. The vector pointing toward the origin from $(1,1)$ is $\mathbf{v} = \langle -1, -1 \rangle$.

The gradient of f is

$$\nabla f = \langle 4x + 5y, \; -6y + 5x \rangle \;\; \Rightarrow \;\; \nabla f(1,1) = \langle 9, -1 \rangle.$$

Therefore

$$D_{\mathbf{v}} f(1,1) = \frac{\nabla f \cdot \mathbf{v}}{|\mathbf{v}|} = \frac{\langle 9, -1 \rangle \cdot \langle -1, -1 \rangle}{\sqrt{(-1)^2 + (-1)^2}} = \frac{-8}{\sqrt{2}} = \boxed{-4\sqrt{2}}.$$

Geometry of the Gradient

Suppose that $\mathbf{v}$ makes an angle of θ with ∇f (at a particular point).

Then

$$D_{\mathbf{v}} f = \frac{\nabla f \cdot \mathbf{v}}{|\mathbf{v}|} = \frac{|\nabla f||\mathbf{v}| \cos \theta}{\mathbf{v}} = |\nabla f| \cos \theta.$$

This is as large as possible when $\theta = 0$. Thus we have the following interpretation of the geometry of ∇f.

Theorem 2

The direction of ∇f is the direction of the greatest increase of f, and $|\nabla f|$ is the rate of change of f in that direction.

Example 3

Let $f(x, y) = x^2 + xy + y^2$. At $(-1, 3)$, in what direction does f increase most rapidly? What is the rate of change of f in that direction?

Solution. We are simply being asked to compute $\nabla f(-1, 3)$ and its magnitude.

Since

$$\nabla f = \langle 2x + y, x + 2y \rangle$$

the greatest rate of change at $(-1, 3)$ occurs in the direction of

$$\nabla f(-1, 3) = \boxed{\langle 1, 5 \rangle}.$$

The rate of change of f in that direction is

$$|\nabla f(-1, 3)| = \sqrt{1^2 + 5^2} = \boxed{\sqrt{26}}.$$

Contours and Gradients

If we move along one of its contours, $f(x, y)$ remains constant.

So if $\mathbf{v}$ is tangent to a contour of $f(x, y)$, then

$$0 = D_{\mathbf{v}} f = \frac{\nabla f \cdot \mathbf{v}}{|\mathbf{v}|} \Rightarrow \nabla f \cdot \mathbf{v} = 0.$$

That is,

$$\boxed{\nabla f \text{ is perpendicular to the contours of } f.}$$

Example 4

Find the tangent line to the curve $x^3 + xy + y^3 = 3$ at $(1, 1)$.

Solution. The given curve is a contour of the function $f(x, y) = x^3 + xy + y^3$.

Since $\nabla f = \langle 3x^2 + y, x + 3y^2 \rangle$, the vector

$$\nabla f(1, 1) = \langle 4, 4 \rangle$$

will be *perpendicular* to $x^3 + xy + y^3 = 3$.

It follows that the tangent line has the equation

$$4(x - 1) + 4(y - 1) = 0 \quad \text{or} \quad \boxed{y = 2 - x}.$$

Gradients in More Variables

We can talk about directional derivatives and gradients in any number of variables, we simply need to include additional components in our computations.

Example 5

Let $T(x, y, z) = \dfrac{3x^2 - 5y}{z}$. At $(1, 2, 3)$, at what rate does T change if we go in the $\mathbf{v} = \langle -1, 0, 1 \rangle$ direction? In what direction does T increase most rapidly at $(1, 2, 3)$? What is the rate of change of T in this direction?

Solution. We begin by computing the gradient:

$$\nabla T = \left\langle \frac{6x}{z}, \frac{-5}{z}, \frac{5y - 3x^2}{z^2} \right\rangle.$$

At $(1, 2, 3)$ we have

$$\nabla T(1, 2, 3) = \left\langle 2, \frac{-5}{3}, \frac{7}{9} \right\rangle.$$

So the rate of change in the $\mathbf{v} = \langle -1, 0, 1 \rangle$ direction is

$$D_{\mathbf{v}} T(1, 2, 3) = \frac{\langle 2, -5/3, 7/9 \rangle \cdot \langle -1, 0, 1 \rangle}{\sqrt{(-1)^2 + 0^2 + 1^2}} = \frac{-2 + \frac{7}{9}}{\sqrt{2}} = \boxed{-\frac{11}{9\sqrt{2}}}.$$

T increases most rapidly in the direction of

$$\boxed{\nabla T(1, 2, 3) = \left\langle 2, \frac{-5}{3}, \frac{7}{9} \right\rangle},$$

with a rate of change given by

$$|\nabla T(1, 2, 3)| = \sqrt{2^2 + \left(\frac{-5}{3}\right)^2 + \left(\frac{7}{9}\right)^2} = \boxed{\frac{\sqrt{598}}{9}}.$$

Example 6

Find the tangent plane to the surface

$$x \sin y + y \sin z + z \sin x = 3$$

at the point $(0, 3\sqrt{2}, \pi/4)$.

Solution. The surface in question is a level surface of

$$F(x, y, z) = x \sin y + y \sin z + z \sin x.$$

In analogy with the two-variable situation, one can show that

$$\nabla F \text{ is orthogonal to level surfaces of } F.$$

This means that we can use $\nabla F(0, 3\sqrt{2}, \pi/4)$ for our normal vector.

We have

$$\nabla F = \langle \sin y + z\cos x, x\cos y + \sin z, y\cos z + \sin x\rangle.$$

Thus

$$\nabla F(0, 3\sqrt{2}, \pi/4) = \left\langle \sin 3\sqrt{2} + \frac{\pi}{4}, \frac{\sqrt{2}}{2}, 3 \right\rangle.$$

The tangent plane therefore has the equation

$$\boxed{\left(\sin 3\sqrt{2} + \frac{\pi}{4}\right)x + \frac{\sqrt{2}}{2}(y - 3\sqrt{2}) + 3\left(z - \frac{\pi}{4}\right) = 0}.$$