The Multivariate Chain Rule

Ryan C. Daileda

Trinity University

Calculus III

Introduction

The *chain rule* in Calculus I tells us how to differentiate *compositions*: functions of the form $f(g(x))$.

A composition can be thought of as starting with a function $f(t)$ and then replacing its variable with another function: $t = g(x)$.

We can form compositions of functions of several variables in an analogous manner, by replacing the given variables with functions of new variables.

The process of finding the partial derivatives of such compositions is called the *Chain Rule*.

Setup. Suppose we are given a function $f(x_1, x_2, \ldots, x_n)$ of n variables, and we replace each x_i with a *function* $x_i(t_1, t_2, \ldots, t_m)$ of new variables $t_1, t_2, \ldots, t_m$.

Question. How are the partial derivatives $\dfrac{\partial f}{\partial x_i}$ and $\dfrac{\partial f}{\partial t_j}$ related?

Let's begin by looking at a concrete example.

Example 1

If $f(x, y) = x^2 + y^3$ and we set $x = t \sin s$, $y = t^4 + s^2$, how are $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$, $\dfrac{\partial f}{\partial s}$ and $\dfrac{\partial f}{\partial t}$ related?

We directly substitute:

$$f(x,y) = x^2 + y^3 = t^2 \sin^2 s + (t^4 + s^2)^3$$

and then compute:

$$\frac{\partial f}{\partial t} = 2t\sin^2 s + 3(t^4+s^2)^2 4t^3 = 2t\sin s \frac{\partial x}{\partial t} + 3(t^4+s^2)^2 \frac{\partial y}{\partial t}$$

$$= 2x\frac{\partial x}{\partial t} + 3y^2\frac{\partial y}{\partial t} = \boxed{\frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}},$$

$$\frac{\partial f}{\partial s} = 2t^2\sin s\cos s + 3(t^4+s^2)^2 2s = 2t\sin s\frac{\partial x}{\partial s} + 3(t^4+s^2)^2\frac{\partial y}{\partial s}$$

$$= 2x\frac{\partial x}{\partial s} + 3y^2\frac{\partial y}{\partial s} = \boxed{\frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}}.$$

□

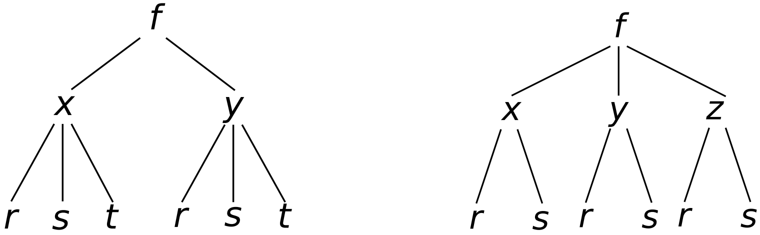
Tree Diagrams

The *chain rule* explains the results of the preceding example.

We start by introducing a *tree diagram*:

- Start with a root vertex labelled f.
- Below f draw branches to vertices labelled with each of the original independent variables.
- Below each of the each independent variables draw branches to each of the new independent variables.

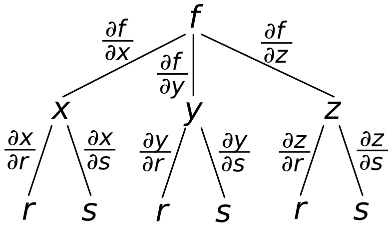
For instance, if f is a function of x and y, and we make x and y both functions of r, s and t, we get the tree on the left.



If f is a function of x, y and z, and we make x, y and z all functions of r and s, we get the tree on the right.

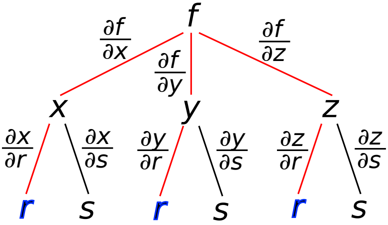
We label each branch in a given tree diagram with the partial derivative of the vertex above with respect to the vertex below.

So in the second example above we get the labelling:



To compute the derivative of f with respect to a variable in the bottom row, we follow every path to that variable, multiplying as we go, and add the results.

Continuing with our example, there are three paths (in red) to the variable r (in blue):



"Multiplying down and adding across" gives the result

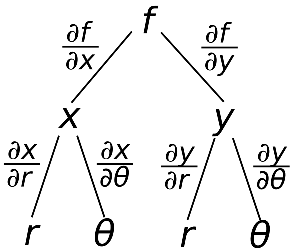
$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r}.$$

Examples

Example 2

If $f(x, y) = x^2y^3$, $x = r\cos\theta$ and $y = r\sin\theta$, use the chain rule to compute $\dfrac{\partial f}{\partial r}$ and $\dfrac{\partial f}{\partial \theta}$.

Solution. We have the following tree diagram:



$\Rightarrow$

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}$$

$$= 2xy^3\cos\theta + 3x^2y^2\sin\theta,$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}$$

$$= -2xy^3 r\sin\theta + 3x^2y^2 r\cos\theta.$$

We now substitute $x = r\cos\theta$ and $y = r\sin\theta$:

$$\frac{\partial f}{\partial r} = 2xy^3\cos\theta + 3x^2y^2\sin\theta$$
$$= \boxed{5r^4\cos^2\theta\sin^3\theta},$$

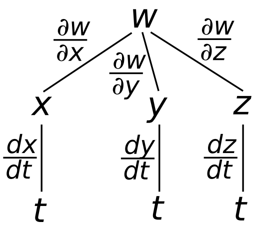
$$\frac{\partial f}{\partial \theta} = -2xy^3 r\sin\theta + 3x^2y^2 r\cos\theta$$
$$= \boxed{r^5(-2\cos\theta\sin^4\theta + 3\cos^3\theta\sin^2\theta)}.$$

□

Example 3

If $w = xe^{y/z}$ and $x = t^2$, $y = 1 - t$ and $z = 1 + 2t$, compute $\dfrac{dw}{dt}$.

Solution. We have the following tree diagram:



$\Rightarrow$

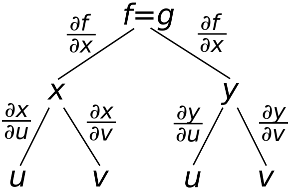
$$\begin{aligned}
\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} \\
&= e^{y/z}(2t) + \frac{x}{z}e^{y/z}(-1) - \frac{xy}{z^2}e^{y/z}(2) \\
&= e^{\frac{1-t}{1+2t}}\left(2t - \frac{t^2}{1+2t} - \frac{2t^2(1-t)}{(1+2t)^2}\right) \\
&= \boxed{e^{\frac{1-t}{1+2t}}\left(\frac{8t^3+5t^2+2t}{(1+2t)^2}\right)}.
\end{aligned}$$

Example 4

Suppose that $g(u, v) = f(e^u + \sin v, e^u + \cos v)$. Given the following table of values, compute $g_u(0, 0)$ and $g_v(0, 0)$.

	f	g	f_x	f_y
$(0, 0)$	3	6	4	8
$(1, 2)$	6	3	2	5

Solution. We have $f = f(x, y)$ with $x = e^u + \sin v$ and $y = e^u + \cos v$. This yields the following diagram:



$\Rightarrow$

$$\frac{\partial g}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$

Thus

$$\begin{aligned} g_u(u,v) &= f_x(x,y)e^u + f_y(x,y)e^u \\ &= f_x(e^u+\sin v, e^u+\cos v)e^u + f_y(e^u+\sin v, e^u+\cos v)e^u, \end{aligned}$$

$$\begin{aligned} g_v(u,v) &= f_x(x,y)\cos v - f_y(x,y)\sin v \\ &= f_x(e^u+\sin v, e^u+\cos v)\cos v - f_y(e^u+\sin v, e^u+\cos v)\sin v, \end{aligned}$$

so that, according to the table,

$$\begin{aligned} g_u(0,0) &= f_x(e^0+\sin 0, e^0+\cos 0)e^0 + f_y(e^0+\sin 0, e^0+\cos 0)e^0 \\ &= f_x(1,2) + f_y(1,2) = 2+5 = \boxed{7}, \end{aligned}$$

$$\begin{aligned} g_u(0,0) &= f_x(e^0+\sin 0, e^0+\cos 0)\cos 0 - f_y(e^0+\sin 0, e^0+\cos 0)\sin 0 \\ &= f_x(1,2) = \boxed{2}. \end{aligned}$$

□

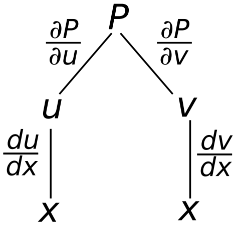
Example 5

Use the Multivariate Chain Rule to derive the ordinary Product and Quotient Rules from Calculus I.

Solution. Given $f(x)$ and $g(x)$, let $P(u,v) = uv$ and $Q(u,v) = u/v$.

Then $P(f(x), g(x)) = f(x)g(x)$ and $Q(f(x), g(x)) = \frac{f(x)}{g(x)}$.

The chain rule gives:



$\Rightarrow$

$$\begin{aligned}\frac{d}{dx}(f(x)g(x)) = \frac{dP}{dx} &= \frac{\partial P}{\partial u}\frac{du}{dx} + \frac{\partial P}{\partial v}\frac{dv}{dx} \\ &= v\frac{df}{dx} + v\frac{dg}{dx} \\ &= g(x)\frac{df}{dx} + f(x)\frac{dg}{dx}.\end{aligned}$$

Similarly,

$$
\begin{aligned}
\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{dQ}{dx} &= \frac{\partial Q}{\partial u}\frac{du}{dx} + \frac{\partial Q}{\partial v}\frac{dv}{dx} \\
&= \frac{1}{v}\frac{df}{dx} - \frac{u}{v^2}\frac{dg}{dx} \\
&= \frac{1}{g(x)}\frac{df}{dx} - \frac{f(x)}{g(x)^2}\frac{dg}{dx} \\
&= \frac{g(x)\frac{df}{dx} - f(x)\frac{dg}{dx}}{g(x)^2}.
\end{aligned}
$$

$\square$

Implicit Differentiation

Given a differentiable function $f(x, y)$, the *Implicit Function Theorem* states that the implicit curve with equation

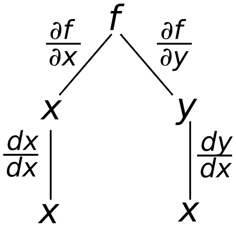
$$f(x, y) = 0$$

defines y as a differentiable function of x at every point where $f_y \neq 0$.

In Calculus I you learn the process of *implicit differentiation* for computing $\frac{dy}{dx}$.

If we use the Chain Rule we can derive a much faster procedure involving the partial derivatives of f.

Setting $y = y(x)$ and differentiating both sides of $f(x, y) = 0$ with respect to x, the chain rule gives:



$$\Rightarrow \qquad \begin{aligned} 0 = \frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} \end{aligned}$$

Solving for $\frac{dy}{dx}$ we obtain

$$\boxed{\frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial y}}.$$

Example 6

If $x^3 + x^2y + y^2x + y^3 = 0$, find $\frac{dy}{dx}$.

Solution. If we set $f(x,y) = x^3 + x^2y + y^2x + y^3$, then according to the preceding formula,

$$\frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial y} = \boxed{\frac{-(3x^2 + 2xy + y^2)}{x^2 + 2xy + 3y^2}}.$$

□

In a similar way, given a differentiable function $F(x, y, z)$, the implicit surface defined by

$$F(x, y, z) = 0$$

defines z as a function of x and y wherever $F_z \neq 0$.

Setting $z = z(x, y)$ and applying the Chain Rule we obtain

$$\begin{aligned} 0 = \frac{\partial F}{\partial x} &= \frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} \\ &= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x}. \end{aligned}$$

Hence

$$\boxed{\frac{\partial z}{\partial x} = \frac{-\partial F/\partial x}{\partial F/\partial z}}.$$

Likewise one can show that

$$\boxed{\frac{\partial z}{\partial y} = \frac{-\partial F/\partial y}{\partial F/\partial z}}.$$

Example 7

If $x^2y + y^2z + z^2x = xyz$, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

Solution. The given equation is equivalent to

$$F(x, y, z) = x^2y + y^2z + z^2x - xyz = 0.$$

Thus

$$\frac{\partial z}{\partial x} = \frac{-\partial F/\partial x}{\partial F/\partial z} = \boxed{\frac{-(2xy + z^2 - yz)}{y^2 + 2xz - xy}}.$$

And

$$\frac{\partial z}{\partial y} = \frac{-\partial F/\partial y}{\partial F/\partial z} = \boxed{\frac{-(x^2 + 2yz - xz)}{y^2 + 2xz - xy}}.$$