Absolute/Global Extrema

Ryan C. Daileda

Trinity University

Calculus III

Global Extrema

Definition

Given $f(x, y)$ with domain $D \subset \mathbb{R}^2$, we say:

- f has an *absolute/global maximum* at $(a, b) \in D$ if $f(a, b) \geq f(x, y)$ for *all* $(x, y) \in D$.
- f has an *absolute/global minimum* at $(a, b) \in D$ if $f(a, b) \leq f(x, y)$ for *all* $(x, y) \in D$.

Remark. The existence of global extrema depends as much on $f(x, y)$ as it does on D.

Example

Consider the function $f(x, y) = \sqrt{x^2 + y^2}$ (the distance from $(0, 0)$ to (x, y)).

1. If $D = \mathbb{R}^2$, then:
   - f has a global minimum at $(0, 0)$;
   - f has *no* global maximum.
2. If $D = \{(x, y) \,|\, x^2 + y^2 \leq 1\}$ (the unit disk), then:
   - f has a global minimum at $(0, 0)$;
   - f has global maxima all along $x^2 + y^2 = 1$.

3. If $D = \{(x, y) \,|\, 1 < x^2 + y^2 < 4\}$, then f does not have *any* global extrema.

We would like conditions on $f(x, y)$ and its domain D that guarantee the existence of global extrema.

Recall. The *Extreme Value Theorem* in Calc. I states that if $f(x)$ is *continuous* on the *closed interval* $[a, b]$, then f has global extrema there.

We have already discussed continuity in multiple variables.

Question. What is the analogue of a closed interval in $\mathbb{R}^2$?

Compact Sets

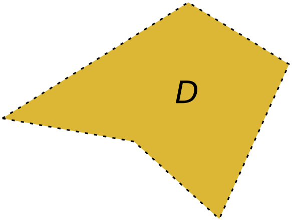
Definition

A set $D \subset R^2$ is:

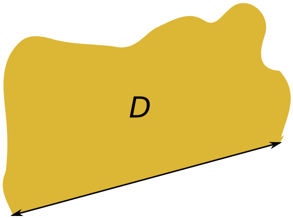
- *bounded* if it can be enclosed in a circle;
- *closed* if it includes its boundary (denoted by ∂D);

We say D is *compact* if it is both closed and bounded.

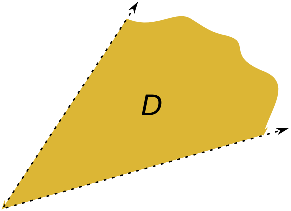
Examples.



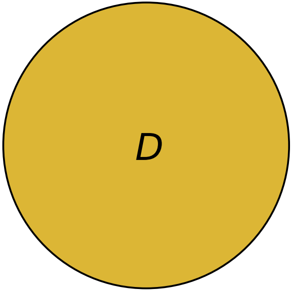
Bounded, not closed



Closed, not bounded



Not bounded, not closed



Compact

Compact subsets of $\mathbb{R}^2$ will play the role that closed intervals in $\mathbb{R}$ play in Calculus I.

We can now state our main result.

Theorem 1 (Extreme Value Theorem)

If $f(x, y)$ is continuous on a compact set $D \subset \mathbb{R}^2$, then f has global extrema there.

The EVT guarantees the existence of global extrema, but how do we find them?

Theorem 2

If $f(x, y)$ is differentiable on a compact set $D \subset \mathbb{R}^2$, then the extrema of f must *occur:*

- *At critical points within D;* OR
- *On the boundary ∂D.*

Recall that if $f(x)$ is differentiable on $[a, b]$, then the extrema of f occur:

- At the critical points of f in (a, b); OR
- At $x = a$ or $x = b$ (the *boundary* points).

This yields the following one variable optimization procedure:

1. Find the critical points of f in (a, b).
2. Includes the boundary points $x = a$ and $x = b$.
3. Compare values.

2D Optimization

We have the following analogous optimization procedure for $f(x, y)$ on a compact domain D:

1. Find the critical points of f in D.
2. Find the extrema of f on ∂D.
3. Compare values.

The "new" step here is finding the extrema along the boundary.

For now we will proceed by ad hoc means, finding *some* way to reduce to a single variable optimization problem.

Example 1

Find the global extrema of $f(x, y) = x^3 + 4y^2 - 3x$ on the closed disk $x^2 + y^2 \leq 2$.

Solution. Because the domain is compact and f is continuous, the EVT guarantees the existence of global extrema.

Critical Points. We set

$$f_x = 3x^2 - 3 = 0 \;\Rightarrow\; x = \pm 1,$$

$$f_y = 8y = 0 \;\Rightarrow\; y = 0,$$

which yields the two critical points $(\pm 1, 0)$, both of which are in our domain.

Boundary. The boundary of the disk is the circle $x^2 + y^2 = 2$, centered at the origin, with radius $\sqrt{2}$.

Since $y^2 = 2 - x^2$ on the boundary, we find that

$$\begin{aligned} f(x, y) &= x^3 + 4y^2 - 3x \\ &= x^3 + 4(2 - x^2) - 3x \\ &= x^3 - 4x^2 - 3x + 8 = g(x). \end{aligned}$$

And as we move along the circle, x varies from $-\sqrt{2}$ to $\sqrt{2}$.

So we need to optimize $g(x)$ on $[-\sqrt{2}, \sqrt{2}]$.

First we find the critical points of g:

$$g'(x) = 3x^2 - 8x - 3 = (3x+1)(x-3) = 0 \quad \Rightarrow \quad x = 3, -1/3.$$

Only $x = -1/3$ is in the interval $[-\sqrt{2}, \sqrt{2}]$.

Now we evaluate g at the one critical point *and* the endpoints:

x	$g(x)$
$-1/3$	$230/27 \approx 8.5$
$\sqrt{2}$	$-\sqrt{2} \approx -1.4$
$-\sqrt{2}$	$\sqrt{2} \approx 1.4$

And we compare to the values of f at its critical points:

(x,y)	$f(x,y)$
$(1,0)$	-2
$(-1,0)$	2

We conclude that f has the absolute maximum value

$$f\left(-\frac{1}{3}, \pm\frac{\sqrt{17}}{3}\right) = \frac{230}{27}$$

on the boundary, and has the absolute minimum value

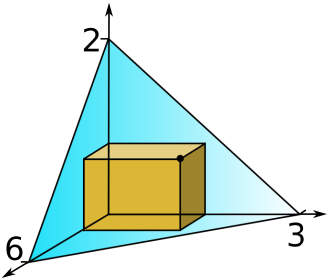
$$f(1,0) = -2$$

in the interior. □

Remark. We found the y-coordinates of the boundary points by using the relationship $x^2 + y^2 = 2$.

Example 2

A rectangular box has three faces in the coordinate planes and one vertex on the plane $x + 2y + 3z = 6$, as shown below. Find the largest possible volume of such a box.



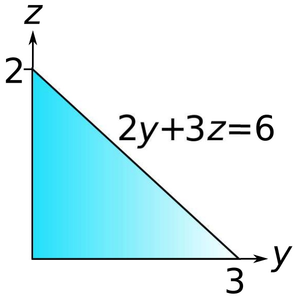
Solution. Let (x, y, z) be the coordinates of the vertex on the plane.

The volume of the box is then

$$\begin{aligned} V &= xyz = (6 - 2y - 3z)yz \\ &= 6yz - 2y^2z - 3yz^2 \end{aligned}$$

To find the domain of V we project the portion of the plane $x + 2y + 3z = 6$ that lies in the first octant onto the yz-plane.

This results in the triangular region shown below:



Along the boundary we have

$$V = (6 - 2y - 3z)yz = 0.$$

So the maximum of V *must* occur at a critical point in the interior.

The critical points of V are given by

$$V_y = 6z - 4yz - 3z^2 = z(6 - 4y - 3z) = 0,$$
$$V_z = 6y - 2y^2 - 6yz = 2y(3 - y - 3z) = 0.$$

On the interior of the domain $y, z \neq 0$. Thus we must have

$$\left.\begin{array}{r} 6 - 4y - 3z = 0 \\ 3 - y - 3z = 0 \end{array}\right\} \Rightarrow \quad y = 1, \;\; z = 2/3.$$

Since there is only one critical point, it immediately follows that the maximum volume is

$$V(1, 2/3) = (6 - 2 \cdot 1 - 3 \cdot \frac{2}{3}) \cdot 1 \cdot \frac{2}{3} = \boxed{\frac{4}{3}}.$$

□

Remark. The dimensions of the box can be found using the relationship $x + 2y + 3z = 6$. They are $2 \times 1 \times 2/3$.

Example 3

Find the global extrema of $f(x, y) = 16xy^2 + 8xy - 15x + 16x^2$ on the set $D = \{(x, y) : |x| \leq 1, |y| \leq 1\}$.

Solution. The set D is the square bounded by the four lines $x = \pm 1$ and $y = \pm 1$.

Since D is compact and f is continuous, the EVT guarantees the existence of global extrema.

Critical Points. We set

$$f_x = 16y^2 + 8y - 15 + 32x = 0,$$
$$f_y = 32xy + 8x = 8x(4y + 1) = 0.$$

The second equation implies that $x = 0$ or $y = -1/4$.

$\underline{x = 0}$: In this case the first equation becomes

$$16y^2 + 8y - 15 = 0 \;\Leftrightarrow\; (4y-3)(4y+5) = 0 \;\Leftrightarrow\; y = -5/4, 3/4.$$

This yields the critical points $(0, 3/4)$ and $(0, -5/4)$, only the first of which is in D.

$\underline{y = -1/4}$: In this case the first equation becomes

$$1 - 2 - 15 + 32x = 0 \;\Leftrightarrow\; 32x = 16 \;\Leftrightarrow\; x = 1/2.$$

This yields the critical point $(1/2, -1/4)$, which is in D.

Boundary. Now we optimize f on each of the edges.

Top: This is given by $y = 1$, $-1 \leq x \leq 1$. Here

$$f(x,y) = f(x,1) = 9x + 16x^2.$$

Applying the usual Calc. I optimization procedure on the interval we find:

$$\text{max. is } f(1,1) = 25,$$
$$\text{min. is } f\left(-\frac{9}{32},1\right) = -\frac{81}{64}.$$

Bottom: This is given by $y = -1$, $-1 \le x \le 1$. Here

$$f(x,y) = f(x,-1) = -7x + 16x^2,$$

and

$$\text{max. is } f(-1,-1) = 23,$$
$$\text{min. is } f\left(\frac{7}{32},-1\right) = -\frac{49}{64}.$$

Right Side: This is given by $x = 1$, $-1 \le y \le 1$. Here

$$f(x,y) = f(1,y) = 16y^2 + 8y + 1,$$

and

$$\text{max. is } f(1,1) = 25,$$
$$\text{min. is } f\left(1,-\frac{1}{4}\right) = 0.$$

Left Side: This is given by $x = -1$, $-1 \le y \le 1$. Here

$$f(x, y) = f(-1, y) = -16y^2 - 8y + 31,$$

and

$$\text{max. is } f\left(-1, -\frac{1}{4}\right) = 32,$$
$$\text{min. is } f(-1, 1) = 7.$$

Taking into account the values at the critical points, which are

$$f(0, 3/4) = 0 \quad \text{and} \quad f(1/2, -1/4) = -4,$$

we find that:

$$\text{Absolute Maximum Value is } f(-1, -1/4) = 32,$$
$$\text{Absolute Minimum Value is } f(1/2, -1/4) = -4.$$

Graph of $f(x, y)$