The Method of Lagrange Multipliers

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Calculus III

Recall

To find the extrema of a differentiable function $f(x, y)$ on a compact domain $D \subset \mathbb{R}^2$:

1. Determine the critical points of f inside D.
2. Determine the extrema of f on the boundary ∂D.
3. Compare the values of f at the points found in **1** and **2**.

Step **2** can be quite challenging.

However, if ∂D is given by an equation, we can appeal to the *Method of Lagrange Multipliers*.

Motivating Example

Example 1

Find the global extrema of $f(x, y) = x^2 + xy + y^2$ on the circle $x^2 + y^2 = 4$.

See Maple diagram.

Solution 1. On the circle we have $y = \pm\sqrt{4 - x^2}$, $-2 \leq x \leq 2$.

So we need to optimize the *pair* of functions

$$f(x, \pm\sqrt{4 - x^2}) = x^2 \pm x\sqrt{4 - x^2} + (4 - x^2) = 4 \pm x\sqrt{4 - x^2}$$

on the closed interval $[-2, 2]$.

The product and chain rules give

$$\frac{d}{dx}\left(4 \pm x\sqrt{4-x^2}\right) = \pm\left(\sqrt{4-x^2} - \frac{x^2}{\sqrt{4-x^2}}\right)$$
$$= \pm\frac{4-2x^2}{\sqrt{4-x^2}}$$

which vanishes iff $x = \pm\sqrt{2}$.

We then have the table of values

x	$4 \pm x\sqrt{4-x^2}$
$\pm\sqrt{2}$	2, 6
± 2	4

We find that the absolute maximum value is 6 and the absolute minimum is 2. □

Solution 2. The circle can be *parametrized* by

$$x = 2\cos\theta,$$
$$y = 2\sin\theta,$$

with $0 \leq \theta \leq 2\pi$.

Substitution yields

$$\begin{aligned} f(2\cos\theta, 2\sin\theta) &= 4\cos^2\theta + 4\cos\theta\sin\theta + 4\sin^2\theta \\ &= 4(1+\cos\theta\sin\theta) = g(\theta). \end{aligned}$$

To find the critical points we set

$$g'(\theta) = 4(\cos^2\theta - \sin^2\theta) = 4\cos 2\theta = 0,$$

and find that $\theta = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$.

Including the endpoint(s) we get the table of values

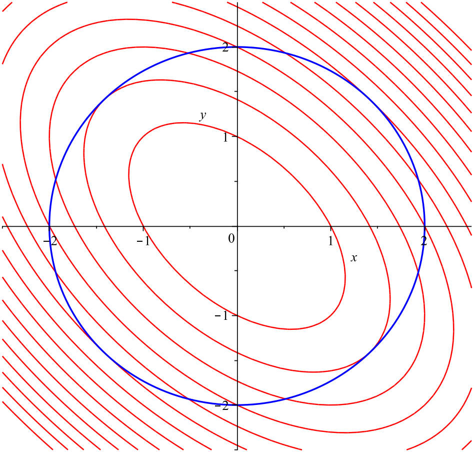
θ	$4(1+\cos\theta\sin\theta)$
$\pi/4, 5\pi/4$	6
$3\pi/4, 7\pi/4$	2
$0, 2\pi$	4

Again we find the absolute maximum and minimum values of 6 and 2 (resp.). □

Remark. Note that the extreme values occur at the four "corners" of the circle $x^2 + y^2 = 4$.

A Better Idea

Solution 3. Let's look at a contour plot of $f(x,y)$ and superimpose the circle $x^2+y^2=4$.



If $\mathbf{v}$ is tangent to the circle $x^2 + y^2 = 4$, the extrema of $f(x, y)$ must occur where

$$\begin{aligned} 0 = D_{\mathbf{v}} f = \frac{\nabla f \cdot \mathbf{v}}{|\mathbf{v}|} &\iff \nabla f \text{ is orth. to } \mathbf{v} \\ &\iff \nabla f \text{ is normal to } \underbrace{x^2 + y^2}_{g(x,y)} = 4 \\ &\iff \nabla f \text{ is parallel to } \nabla g \\ &\iff \nabla f = \lambda \nabla g \end{aligned}$$

for some scalar λ. That is,

$$\langle 2x + y, x + 2y \rangle = \lambda \langle 2x, 2y \rangle.$$

This gives us the system of three equations

$$\begin{aligned} 2x + y &= 2\lambda x, \\ x + 2y &= 2\lambda y, \\ x^2 + y^2 &= 4, \end{aligned}$$

in the three unknowns x, y and λ.

The *Lagrange Multiplier* λ is an auxiliary quantity, so we eliminate it.

Multiply the first equation by y and the second by x to get

$$\left.\begin{aligned} 2xy + y^2 &= 2\lambda xy \\ x^2 + 2xy &= 2\lambda xy \end{aligned}\right\} \;\Rightarrow\; 2xy + y^2 = x^2 + 2xy \;\Rightarrow\; x^2 = y^2.$$

Plugging this into the third equation (the *constraint*) gives

$$2x^2 = 4 \;\Rightarrow\; x^2 = 2 \;\Rightarrow\; x = \pm\sqrt{2}.$$

Since $x^2 = y^2$, we get $y = \pm\sqrt{2}$, with no correlation between the signs.

So we have 4 *Lagrange points*: $(\pm\sqrt{2}, \sqrt{2})$, $(\pm\sqrt{2}, -\sqrt{2})$. And the extrema *must* occur here.

Finally, we construct a table of values:

(x, y)	$x^2 + xy + y^2$
$(\sqrt{2}, \sqrt{2})$	6
$(\sqrt{2}, -\sqrt{2})$	2
$(-\sqrt{2}, \sqrt{2})$	2
$(-\sqrt{2}, -\sqrt{2})$	6

$\Rightarrow$ Abs. Max. Value is 6,
Abs. Min. Value is 2.

Constrained Optimization

Let's take a look at the general situation.

Suppose that $f(x, y)$ and $g(x, y)$ are differentiable and that $\nabla g \neq \mathbf{0}$ along the contour $g(x, y) = k$.

We would like to optimize $f(x, y)$ subject to the *constraint* $g(x, y) = k$.

The *Implicit Function Theorem* implies that the equation $g(x, y) = k$ defines y as a differentiable function of x (or vice versa).

This allows us to reduce to a single variable problem.

For convenience, assume $y = y(x)$.

Recall that along $g(x, y) = k$ the chain rule yields

$$\frac{dy}{dx} = \frac{-\partial g/\partial x}{\partial g/\partial y}.$$

So we have $f(x, y) = f(x, y(x))$. The critical points of this function occur where

$$0 = \frac{d}{dx} f(x, y(x)) = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} = \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\frac{\partial g/\partial x}{\partial g/\partial y}$$

Multiplying through by $\partial g/\partial y$ this becomes

$$0 = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} = f_x g_y - f_y g_x = \begin{vmatrix} f_x & f_y \\ g_x & g_y \end{vmatrix} = \nabla f \times \nabla g,$$

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This means that ∇f and ∇g are parallel, i.e. $\nabla f = \lambda \nabla g$ for some scalar λ. This proves:

Theorem 1

Suppose $f(x, y)$ and $g(x, y)$ are differentiable and that $\nabla g \neq \mathbf{0}$ along the contour $g(x, y) = k$. The extrema of $f(x, y)$ subject to the constraint $g(x, y) = k$ (if they exist) must occur where $\nabla f = \lambda \nabla g$.

Remarks.

- We will call the points where $\nabla f = \lambda \nabla g$ the *Lagrange points* of the constrained optimization problem.
- The quantity λ is a *Lagrange multiplier*. In practical applications it has meaningful interpretations.

More Remarks

- If f and g are differentiable functions of n of variables, and $\nabla g \neq \mathbf{0}$ along the level set $g = k$ (an $(n-1)$-dimensional *manifold*), a similar argument shows that the extrema of f subject to $g = k$ must occur where $\nabla f = \lambda \nabla g$.
- By equating components, the vector equation $\nabla f = \lambda \nabla g$ yields n ordinary equations in $n+1$ variables.
- Therefore the *Lagrange system*

$$\nabla f = \lambda \nabla g,$$
$$g = k,$$

amounts to $n+1$ equations in $n+1$ unknowns.

- We will not utilize λ, so our goal is primarily to eliminate it.
- In 2 and 3 variables this is easily achieved by replacing $\nabla f = \lambda \nabla g$ with the equivalent equation $\nabla f \times \nabla g = \mathbf{0}$.

Example 2

Find the global extrema of $f(x, y) = 2x^2 + y^2$ subject to the constraint $x^2 + xy + y^2 = 2$.

Solution. $f(x, y)$ is continuous and $g(x, y) = x^2 + xy + y^2 = 2$ represents an ellipse, which is compact.

The EVT guarantees that f has global extrema subject to the constraint $g = 2$.

The Lagrange equations are

$$\begin{aligned}\nabla f \times \nabla g &= \begin{vmatrix} f_x & f_y \\ g_x & g_y \end{vmatrix} = f_x g_y - f_y g_x \\ &= 4x(x+2y) - 2y(2x+y) \\ &= 4x^2 + 4xy - 2y^2 = 0, \\ x^2 + xy + y^2 &= 2.\end{aligned}$$

If we multiply the second equation by 4 and subtract it from the first we obtain

$$-6y^2 = -8 \;\Rightarrow\; y^2 = \frac{4}{3} \;\Rightarrow\; y = \pm\frac{2\sqrt{3}}{3}.$$

Substituting $y = \pm 2\sqrt{3}/3$ back into $x^2 + xy + y^2 = 2$ gives us

$$x = \pm\left(1 \pm \frac{\sqrt{3}}{3}\right),$$

with no correlation between the signs.

This means we have 4 Lagrange points with the following values:

(x, y)	$2x^2 + y^2$
$\left(\mp\left(1+\frac{\sqrt{3}}{3}\right), \pm\frac{2\sqrt{3}}{3}\right)$	$4 + \frac{4\sqrt{3}}{3}$
$\left(\pm\left(1-\frac{\sqrt{3}}{3}\right), \pm\frac{2\sqrt{3}}{3}\right)$	$4 - \frac{4\sqrt{3}}{3}$

So the absolute maximum value is $4 + 4\sqrt{3}/3 \approx 6.309$ and the absolute minimum value is $4 - 4\sqrt{3}/3 \approx 1.691$. □

More Examples

Example 3

Find the global extrema of $f(x, y, z) = 2x + 6y + 10z$ on the sphere $x^2 + y^2 + z^2 = 35$.

Solution. Since f is continuous and the sphere $g(x, y, z) = x^2 + y^2 + z^2 = 35$ is compact, the EVT guarantees global extrema exist.

The Lagrange system is

$$\begin{aligned} 2 &= 2\lambda x, \\ 6 &= 2\lambda y, \\ 10 &= 2\lambda z, \\ 35 &= x^2 + y^2 + z^2. \end{aligned}$$

Since $\lambda \neq 0$ (why?), we can solve the first three equations for $1/\lambda$:

$$\frac{1}{\lambda} = x = \frac{y}{3} = \frac{z}{5} \ \Rightarrow \ y = 3x \text{ and } z = 5x.$$

Plugging these into the sphere's equation we get

$$35 = x^2 + y^2 + z^2 = x^2 + (3x)^2 + (5x)^2 = 35x^2 \ \Rightarrow \ x = \pm 1.$$

We therefore have two Lagrange points with values:

(x, y, z)	$2x + 6y + 10z$
$(1, 3, 5)$	70
$(-1, -3, -5)$	-70

$\Rightarrow$ Abs. Max. Value is 70,
Abs. Min. Value is -70.

□

Example 4

Recall the problem of maximizing the volume $V = xyz$ of a box with one vertex on the plane $x + 2y + 3z = 6$. Use Lagrange multipliers to find the maximum volume.

Solution. With $g(x, y, z) = x + 2y + 3z$ the Lagrange system is

$$\begin{aligned} yz &= \lambda, \\ xz &= 2\lambda, \\ xy &= 3\lambda, \\ 6 &= x + 2y + 3z, \end{aligned}$$

which means that

$$\lambda = yz = \frac{xz}{2} = \frac{xy}{3}.$$

If $x = 0$, $y = 0$ or $z = 0$, then $V = 0$, which is not the maximum volume.

So we can divide by x, y, z to obtain

$$y = \frac{x}{2} \text{ and } z = \frac{x}{3}.$$

Plugging into $x + 2y + 3z = 6$ gives us

$$6 = x + 2\left(\frac{x}{2}\right) + 3\left(\frac{x}{3}\right) = 3x \;\Rightarrow\; x = 2 \;\Rightarrow\; y = 1 \text{ and } z = \frac{2}{3}.$$

So the optimal dimensions are $2 \times 1 \times 2/3$, with volume $V = 4/3$. □

Multiple Constraints

Suppose we are asked to optimize $f(x, y, z)$ subject to the *two* constraint equations $g(x, y, z) = k$ and $h(x, y, z) = \ell$.

The intersection of the level surfaces $g(x, y, z) = k$ and $h(x, y, z) = \ell$ is a curve C.

Because it lies in both surfaces, C is perpendicular to both of their normal vectors.

Since the normal vectors are ∇g and ∇h, this means that

$$\mathbf{v} = \nabla g \times \nabla h \text{ is tangent to } C.$$

The extrema of f on C therefore occur where

$$\begin{aligned} 0 = D_{\mathbf{v}} f = \frac{\nabla f \cdot \mathbf{v}}{|\mathbf{v}|} \ &\Rightarrow \ \nabla f \cdot \mathbf{v} = 0 \\ &\Rightarrow \ \nabla f \text{ is orth. to } \mathbf{v} = \nabla g \times \nabla h \\ &\Rightarrow \ \nabla f \text{ is in the plane of } \nabla g \text{ and } \nabla h \\ &\Rightarrow \ \nabla f = \lambda \nabla g + \mu \nabla h \end{aligned}$$

for some scalars λ and μ.

Remark. The equivalent condition $\nabla f \cdot (\nabla g \times \nabla h) = 0$ can also be useful.

Example 5

Find the extrema of $f(x, y, z) = x + 2y$ subject to the constraints $x + y + z = 1$ and $y^2 + z^2 = 4$.

Solution. The intersection of the plane $g(x, y, z) = x + y + z = 1$ and the cylinder $h(x, y, z) = y^2 + z^2 = 4$ is an ellipse (in $\mathbb{R}^3$), which is compact.

Since f is continuous, the EVT guarantees the existence of global extrema.

The two constraint Lagrange system is

$$
\begin{aligned}
&1 = \lambda \cdot 1 + \mu \cdot 0 = \lambda, \\
&2 = \lambda \cdot 1 + \mu \cdot 2y = \lambda + 2\mu y, \\
&0 = \lambda \cdot 1 + \mu \cdot 2z = \lambda + 2\mu z, \\
&x + y + z = 1, \\
&y^2 + z^2 = 4.
\end{aligned}
$$

The first equation tells us that $\lambda = 1$. Plugging this into the second two equations we get

$$\left.\begin{array}{r} 2\mu y = 1 \\ 2\mu z = -1 \end{array}\right\} \;\Rightarrow\; 2\mu = \frac{1}{y} = \frac{-1}{z} \;\Rightarrow\; y = -z.$$

Plugging into the fourth equation we obtain

$$1 = x + y + z = x + y - y = x.$$

Plugging into the last equation we find that

$$4 = y^2 + z^2 = 2y^2 \;\Rightarrow\; y = \pm\sqrt{2} \;\Rightarrow\; z = \mp\sqrt{2}.$$

So we have two Lagrange points: $(1, \pm\sqrt{2}, \mp\sqrt{2})$.

Finally, we have the values

$$f(1, \pm\sqrt{2}, \mp\sqrt{2}) = 1 \pm 2\sqrt{2},$$

so that the absolute maximum is $1 + 2\sqrt{2}$ and the absolute minimum is $1 - 2\sqrt{2}$.