Double Integrals over Rectangular Regions

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Calculus III

Introduction

Today we will begin our study of integration in multiple variables.

As we will see, there are a number of different multivariate integrals, depending on the type of object being integrated.

Specifically, we will (eventually) study: double integrals, triple integrals, line integrals, and surface integrals.

To motivate our constructions, we will begin by reviewing how the integral of a single variable is defined.

Single Variable Integrals

Recall. Given a function $f(x)$ on a closed interval $[a, b]$, the integral $\int_a^b f(x)\,dx$ is defined as the limit of Riemann sums.

Specifically, we first subdivide $[a, b]$ into n subintervals:

$$a = x_0 < x_1 < x_2 < \cdots < x_{i-1} < x_i < \cdots < x_{n-1} < x_n = b.$$

We then form the *Riemann sum* $\displaystyle\sum_{i=1}^{n} f(x_i^*)\Delta x_i$, where:

$$x_i^* = \text{ sample point in the subinterval } [x_{i-1}, x_i],$$

$$\Delta x_i = x_i - x_{i-1} \text{ is the length of } [x_{i-1}, x_i].$$

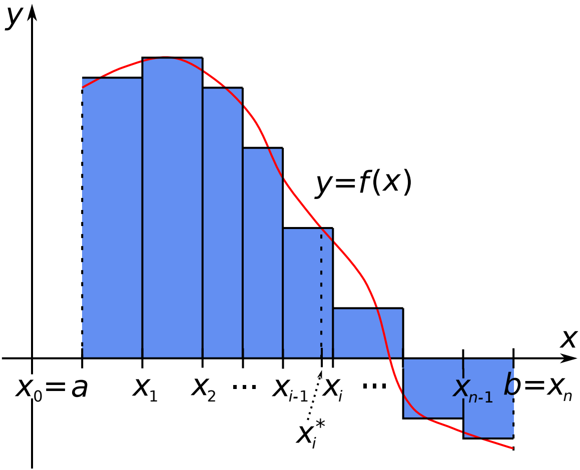
Finally, we take the limit as the size of the subdivisions tends to zero:

$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i,$$

where $\Delta x = \max_{1 \le i \le n} \Delta x_i$ is the maximum size of the subdivisions.

We say that f is *integrable* provided this limit exists (and is independent of the choices made in its construction).

We can visually represent each term $f(x_i^*)\Delta x_i$ in the Riemann sum as the (signed) area of a rectangle that approximately sits between the graph $y = f(x)$ and the x-axis.


y
x
y=f(x)
x0=a
x1
x2
...
xi-1
xi
...
xn-1
b=xn
xi*

The total Riemann sum therefore represents the approximate (signed) area between $y = f(x)$ and the x-axis.

As the rectangles become smaller (i.e. $n \to \infty$), the approximation becomes better and better, and we arrive at the interpretation

$$\int_a^b f(x)\,dx = (\text{Area below } y = f(x) \text{ and above the } x\text{-axis}) - (\text{Area above } y = f(x) \text{ and below the } x\text{-axis})$$

Double Integrals

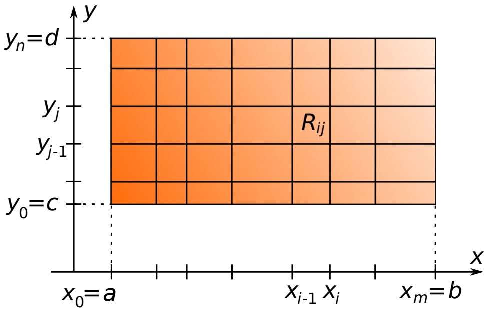
We would like to mimic the procedure above for a function $f(x, y)$ with domain $D \subset \mathbb{R}^2$.

For simplicity, we begin by assuming that $D = R$ is a rectangle with its sides parallel to the coordinate axes:

$$R = [a, b] \times [c, d].$$

We subdivide R into smaller rectangles by simultaneously subdividing the intervals $[a, b]$ and $[c, d]$:

$$a = x_0 < x_1 < x_2 < \cdots < x_{i-1} < x_i < \cdots < x_{m-1} < x_m = b,$$
$$c = y_0 < y_1 < y_2 < \cdots < y_{j-1} < y_j < \cdots < y_{n-1} < y_n = d.$$



We let R_{ij} denote the subrectangle in the (i, j) position. Notice that:

- There are mn total subrectangles R_{ij}.
- The area of R_{ij} is $\Delta A_{ij} = \Delta x_i \Delta y_j = (x_i - x_{i-1})(y_j - y_{j-1})$.

Choosing a sample point (x_i^*, y_j^*) in each R_{ij}, we build the *Riemann sum*

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*)\Delta A_{ij} = \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*)\Delta x_i \Delta y_j.$$

Finally, we let the sizes of the subdivisions tend to zero and define

$$\iint_R f(x,y)\, dA = \lim_{\Delta x, \Delta y \to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*)\Delta A_{ij},$$

where $\Delta x = \max_{1 \leq i \leq m} \Delta x_i$ and $\Delta y = \max_{1 \leq j \leq n} \Delta y_j$.

We say that f is *integrable* provided this limit exists (and is independent of our choices).

Interpreting Double Integrals

The values $f(x_i^*, y_j^*)$ represent heights on the graph $z = f(x, y)$ above the sample points (x_i^*, y_j^*).

The terms $f(x_i^*, y_j^*)\Delta A_{ij}$ of the Riemann sum therefore represent the (signed) *volumes* of rectangular prisms inserted between $z = f(x, y)$ and the xy-plane.

The total Riemann sum therefore represents the approximate (signed) volume between $z = f(x, y)$ and the xy-plane.

See Maple diagram.

It follows that the double integral is a "signed" volume:

$$\iint_R f(x,y)\,dA = (\text{Volume below } z = f(x,y) \text{ and above } R) \\ - (\text{Volume above } z = f(x,y) \text{ and below } R).$$

We can use this interpretation to help us compute the values of double integrals.

Example 1

If $f(x,y) = k$ (constant), evaluate $\displaystyle\iint_R f\,dA$.

Solution. The graph of $z = k$ is a horizontal plane (parallel to the xy-plane).

The region between the graph and R is therefore a rectangular solid.

Thus

$$\iint_R k\, dA = \text{ height } \times \text{ area of } R = \boxed{k \cdot \text{Area}(R)}.$$

□

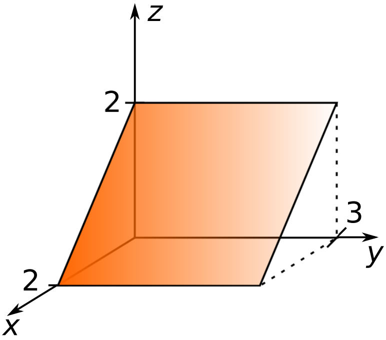
Remark. Compare this to the single variable result

$$\int_a^b k\, dx = k(b-a) = k \cdot \text{Length}([a,b]).$$

Example 2

Compute $\iint_R 2 - x \, dA$, where $R = [0,2] \times [0,3]$.

Solution. The graph $z = 2 - x$ is a plane parallel to the y-axis:



The integral therefore represents the volume of a triangular prism:

$$\iint_R 2 - x\,dA = \text{ area of triangular face } \times \text{ width}$$
$$= \left(\frac{1}{2} \cdot 2 \cdot 2\right) \cdot 3 = \boxed{6}.$$

□

Example 3

Compute $\displaystyle\iint_R 2 - x^2 - y^2\,dA$, where $R = [-1, 1] \times [-1, 1]$.

Solution. The graph $z = 2 - x^2 - y^2$ is a downward opening paraboloid.

The integral therefore represents the volume of a parabolic "tent."

See Maple diagram.

To compute the volume we appeal to *Cavalieri's Principle*, which asserts that volume is the integral of cross-sectional area.

The area of a cross section of the tent perpendicular to the y-axis is given by

$$A(y) = \int_{-1}^{1} 2 - x^2 - y^2 \, dx = 2x - \frac{x^3}{3} - xy^2 \bigg|_{x=-1}^{x=1}$$

$$= \left(2 - \frac{1}{3} - y^2\right) - \left(-2 + \frac{1}{3} + y^2\right) = \frac{10}{3} - 2y^2.$$

The value of the double integral is therefore

$$\iint_R 2-x^2 - y^2\,dA = \int_{-1}^{1} A(y)\,dy = \int_{-1}^{1} \frac{10}{3} - 2y^2\,dy$$

$$= \frac{10}{3}y - \frac{2}{3}y^3\Big|_{-1}^{1} = \left(\frac{10}{3} - \frac{2}{3}\right) - \left(-\frac{10}{3} + \frac{2}{3}\right) = \boxed{\frac{16}{3}}.$$

□

Remark. Notice that if we had simply substituted in the integral expression for $A(y)$, we would have had

$$\iint_R 2 - x^2 - y^2\,dA = \int_{-1}^{1} A(y)\,dy = \int_{-1}^{1}\left(\int_{-1}^{1} 2 - x^2 - y^2\,dy\right)\,dx$$

$$= \int_{-1}^{1}\int_{-1}^{1} 2 - x^2 - y^2\,dy\,dx.$$

Iterated Integrals and Fubini's Theorem

Such an expression is called an *iterated integral*. The following result tells us that double integrals can *always* be expressed as iterated integrals.

Theorem 1 (Fubini)

If $f(x, y)$ is integrable on $R = [a, b] \times [c, d]$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy.$$

Remarks.

- Iterated integrals are always evaluated *inside to outside*.
- Notice that Fubini's theorem tells us that the order of integration *doesn't matter*.

Examples

Remark. Before proceeding to our examples, we note that all continuous functions are necessarily integrable, so that Fubini's theorem automatically applies to their integrals.

Example 4

If $R = [0, \pi/2] \times [0, \pi]$, evaluate $\iint_R \sin(x + 2y)\, dA$.

Solution. By Fubini's theorem we have

$$\iint_R \sin(x + 2y)\, dA = \int_0^{\pi/2} \int_0^{\pi} \sin(x + 2y)\, dy\, dx$$

$$= \int_0^{\pi/2} \frac{-\cos(x + 2y)}{2} \Bigg|_{y=0}^{y=\pi} dx$$

$$= \int_0^{\pi/2} \frac{\cos x}{2} - \frac{\cos(x + 2\pi)}{2} \, dx$$

$$= \int_0^{\pi/2} \frac{\cos x}{2} - \frac{\cos x}{2} \, dx = \int_0^{\pi/2} 0 \, dx = \boxed{0}.$$

□

Example 5

If $R = [-1, 2] \times [1, 3]$, evaluate $\iint_R x^2y + x + y \, dA$.

Solution. By Fubini's theorem we have

$$\iint_R x^2y + x + y \, dA = \int_{-1}^{2} \int_1^3 x^2y + x + y \, dy \, dx$$

$$\int_{-1}^{2}\int_{1}^{3} x^2y + x + y\,dy\,dx = \int_{-1}^{2} \frac{x^2y^2}{2} + xy + \frac{y^2}{2}\Bigg|_{y=1}^{y=3} dx$$

$$= \int_{-1}^{2} \left(\frac{9x^2}{2} + 3x + \frac{9}{2}\right) - \left(\frac{x^2}{2} + x + \frac{1}{2}\right)\,dx$$

$$= \int_{-1}^{2} 4x^2 + 2x + 4\,dx = \frac{4x^3}{3} + x^2 + 4x\Bigg|_{-1}^{2}$$

$$= \left(\frac{32}{3} + 4 + 8\right) - \left(-\frac{4}{3} + 1 - 4\right) = \boxed{27}.$$

□

A function $f(x, y)$ is called *separable* if $f(x, y) = g(x)h(y)$. Notice that in this case

$$\begin{aligned} \int_a^b \int_c^d f(x, y)\, dy\, dx &= \int_a^b \left(\int_c^d g(x)h(y)\, dy \right) dx \\ &= \int_a^b g(x) \left(\int_c^d h(y)\, dy \right) dx \\ &= \left(\int_c^d h(y)\, dy \right) \left(\int_a^b g(x)\, dx \right). \end{aligned}$$

That is, the double integral is simply the product of two single-variable integrals.

This can simplify some double integral computations.

Example 6

Evaluate $\displaystyle\iint_R x^3 e^{y^2}\,dA$, where $R = [-1,1] \times [0,2]$.

Solution. By Fubini's theorem we have

$$\begin{aligned}\iint_R x^3 e^{y^2}\,dA &= \int_{-1}^{1}\int_0^2 x^3 e^{y^2}\,dy\,dx\\ &= \left(\int_{-1}^{1} x^3\,dx\right)\left(\int_0^2 e^{y^2}\,dy\right),\end{aligned}$$

since the integrand is separable. Because x^3 is an odd function and the interval $[-1,1]$ is symmetric about 0,

$$\left(\int_{-1}^{1} x^3\,dx\right)\left(\int_0^2 e^{y^2}\,dy\right) = 0 \cdot \int_0^2 e^{y^2}\,dy = \boxed{0}.$$

Although the order of integration in an iterated integral doesn't matter in principle, in practice it can be rather important.

Example 7

Evaluate the iterated integral $\int_0^1 \int_0^2 xye^{x^2y}\, dy\, dx$.

Solution. If we proceed naïvely and simply integrate by parts in y, we find that

$$\int_0^1 \int_0^2 xye^{x^2y}\, dy\, dx = \int_0^1 \frac{(2x^2-1)e^{2x^2}+1}{x^3}\, dx.$$

This is a rather difficult integral which can be evaluated using power series, for instance.

Because the integrand is continuous, Fubini's theorem, however, guarantees that

$$\int_0^1 \int_0^2 xye^{x^2y}\,dy\,dx = \iint_R xye^{x^2y}\,dA = \int_0^2 \int_0^1 xye^{x^2y}\,dx\,dy,$$

where $R = [0,1] \times [0,2]$.

That is, we are free to reverse the order of integration.

The simple substitution $u = x^2y$ (in x) then yields

$$\int_0^2 \int_0^1 xye^{x^2y}\,dx\,dy = \frac{1}{2}\int_0^2 e^{x^2y}\Big|_{x=0}^{x=1} dy = \frac{1}{2}\int_0^2 e^y - 1\,dy$$

$$= \frac{1}{2}\left(e^y - y\Big|_0^2\right) = \boxed{\frac{e^2-3}{2}}.$$