Double Integrals over General Regions

Ryan C. Daileda

Trinity University

Calculus III

Introduction

In Calculus I we only integrate over closed intervals, because these are the *connected* subsets of $\mathbb{R}$.

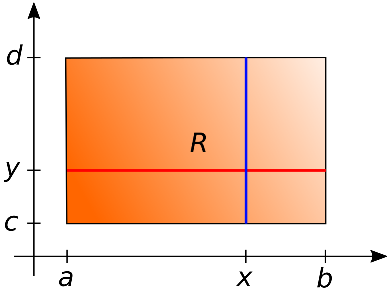
The connected subsets of $\mathbb{R}^2$ aren't nearly as simple, which makes integration in two variables more complicated.

We have seen how to integrate over rectangles whose sides are parallel to the coordinate axes, by thinking in terms of cross sections.

We can apply the same reasoning to express double integrals over more general regions as iterated integrals with *variable* limits.

Fubini's Theorem

Recall. If $f(x,y)$ is integrable on a rectangle $R = [a,b] \times [c,d]$, then:



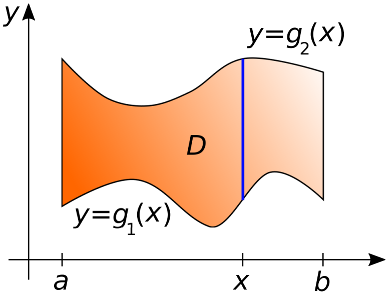
$$\iint_R f(x,y)\,dA$$

$$= \int_a^b \underbrace{\int_c^d f(x,y)\,dy}_{\substack{\text{cross section} \\ \perp \text{ to x-axis}}}\,dx$$

$$= \int_c^d \underbrace{\int_a^b f(x,y)\,dx}_{\substack{\text{cross section} \\ \perp \text{ to y-axis}}}\,dy$$

Type I Regions

A *Type I* region has the form:

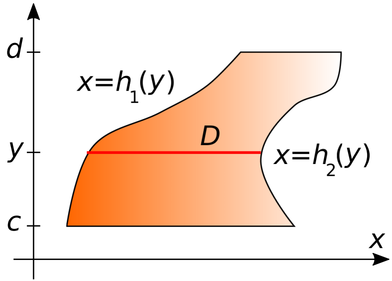


A cross section argument yields:

$$\iint_D f(x,y)\,dA = \int_a^b \underbrace{\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy}_{\substack{\text{cross section}\\ \perp \text{ to x-axis}}}\,dx$$

Type II Regions

A *Type II* region has the form:



A cross section argument yields:

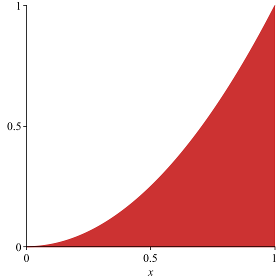
$$\iint_D f(x,y)\,dA$$
$$= \int_c^d \underbrace{\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx}_{\substack{\text{cross section}\\ \perp\text{ to y-axis}}}\,dy$$

Examples

Example 1

Evaluate $\displaystyle\iint_D x\cos y\, dA$, where D is the region bounded by $y = 0$, $y = x^2$ and $x = 1$.

Solution. First we sketch the region D:



This is Type I with "bottom" curve $y = 0$ and "top" curve $y = x^2$.

It is also Type II, with "left side" $x = \sqrt{y}$ and "right side" $x = 1$.

We choose to treat it as Type I to avoid introducing radicals. Thus

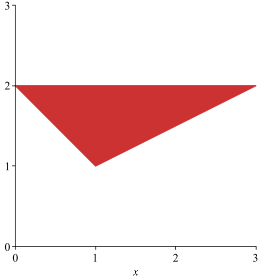
$$\begin{aligned}
\iint_D x\cos y\,dA &= \int_0^1 \int_0^{x^2} x\cos y\,dy\,dx \\
&= \int_0^1 x\sin y\Big|_{y=0}^{y=x^2}\,dx \\
&= \int_0^1 x\sin x^2\,dx \\
&= \frac{-\cos(x^2)}{2}\Big|_0^1 = \boxed{\frac{1-\cos 1}{2}}.
\end{aligned}$$

□

Example 2

Evaluate $\iint y^3\,dA$, where D is the triangular region with vertices $(0,2)$, $(1,1)$ and $(3,2)$.

Solution. First we sketch the region D:



This is Type II with "left side" $x = 2 - y$ and "right side" $x = 2y - 1$.

It is also Type I, but the "bottom" curve is defined piecewise.

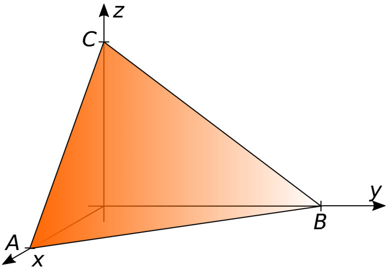
To avoid splitting the integral, we therefore choose to treat D as a Type II region:

$$\begin{aligned}
\iint_D y^3\,dA &= \int_1^2 \int_{2-y}^{2y-1} y^3\,dx\,dy \\
&= \int_1^2 xy^3 \Big|_{x=2-y}^{x=2y-1} dy \\
&= \int_1^2 y^3(2y-1-(2-y))\,dy \\
&= 3\int_1^2 y^4 - y^3\,dy = 3\left(\frac{y^5}{5} - \frac{y^4}{4}\Big|_1^2\right) \\
&= 3\left(\frac{32}{5} - \frac{16}{4} - \frac{1}{5} + \frac{1}{4}\right) = \boxed{\frac{147}{20}}.
\end{aligned}$$

Example 3

Find the volume of the tetrahedron with vertices $(A, 0, 0)$, $(0, B, 0)$, $(0, 0, C)$ and $(0, 0, 0)$ (with $A, B, C > 0$).

Solution. First let's sketch the tetrahedron:

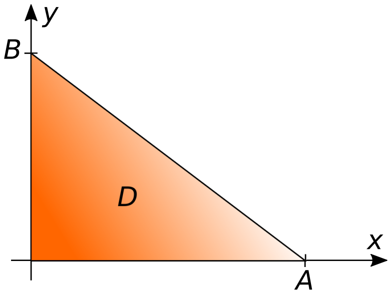


We want the volume under the plane and above the bottom triangular face.

By guess-and-check (or otherwise) we find that the plane is given by

$$\frac{x}{A} + \frac{y}{B} + \frac{z}{C} = 1.$$

So we must compute $\iint_D C\left(1-\frac{x}{A}-\frac{y}{B}\right)\,dA$ where D is:



The line is given by setting $z=0$ in the plane equation:

$$\frac{x}{A}+\frac{y}{B}=1.$$

This is Type I with "bottom" curve $y = 0$ and "top" curve $y=B\left(1-\frac{x}{A}\right)$.

So the volume is

$$C\int_0^A\int_0^{B(1-x/A)} 1-\frac{x}{A}-\frac{y}{B}\,dy\,dx$$

$$=C\int_0^A y-\frac{xy}{A}-\frac{y^2}{2B}\bigg|_{y=0}^{y=B(1-x/A)}\,dx$$

$$=C\int_0^A B\left(1-\frac{x}{A}\right)-\frac{x}{A}B\left(1-\frac{x}{A}\right)-\frac{1}{2B}B^2\left(1-\frac{x}{A}\right)^2\,dx$$

$$=\frac{BC}{2}\int_0^A\left(1-\frac{x}{A}\right)^2\,dx=\frac{BC}{2}\left(\frac{-A}{3}\right)\left(1-\frac{x}{A}\right)^3\bigg|_0^A$$

$$=\boxed{\frac{ABC}{6}}.$$

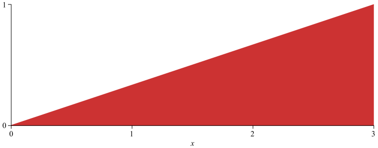
$\square$

Example 4

Evaluate $\displaystyle\int_0^1 \int_{3y}^3 e^{x^2}\,dx\,dy$.

Solution. We *cannot* integrate in the order given since e^{x^2} does not have an elementary antiderivative.

However, the iterated integral is equal to $\displaystyle\iint_D e^{x^2}\,dA$, where D is:



The line on the top/left is $x = 3y$ or $y = x/3$.

Treating this instead as a Type I integral we have

$$\int_0^1 \int_{3y}^3 e^{x^2}\,dx\,dy = \int_0^3 \int_0^{x/3} e^{x^2}\,dy\,dx$$

$$= \frac{1}{3}\int_0^3 xe^{x^2}\,dx$$

$$= \frac{1}{6}e^{x^2}\Big|_0^3 = \boxed{\frac{e^9-1}{6}}.$$

$\square$

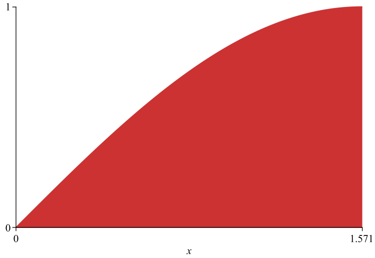
Remark. Note that although it is "impossible" to integrate e^{x^2} with respect to x, it is *trivial* to integrate it with respect to y!

Example 5

Evaluate $\displaystyle\int_0^1 \int_{\arcsin y}^{\frac{\pi}{2}} \cos x \sqrt{1+\cos^2 x}\, dx\, dy$.

Solution. Once again we are faced with an "impossible" Type II integral.

So we sketch the region of integration and reverse the order:



The top/left curve is $x = \arcsin y$ or $y = \sin x$.

The right edge is $x = \pi/2$.

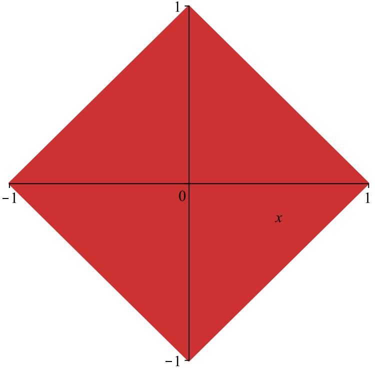
When expressed as a Type I integral we have

$$\begin{aligned}
\int_0^1 \int_{\arcsin y}^{\frac{\pi}{2}} \cos x\sqrt{1+\cos^2 x}\,dx\,dy \\
&= \int_0^{\pi/2}\int_0^{\sin x} \cos x\sqrt{1+\cos^2 x}\,dy\,dx \\
&= \int_0^{\pi/2} \sin x\cos x\sqrt{1+\cos^2 x}\,dx \\
\begin{pmatrix} u = 1+\cos^2 x \\ du = -2\sin x\cos x\,dx \end{pmatrix} \quad &= -\frac{1}{2}\int_2^1 \sqrt{u}\,du \\
&= \frac{1}{2}\int_1^2 \sqrt{u}\,du = \frac{1}{2}\frac{u^{3/2}}{3/2}\bigg|_1^2 \\
&= \boxed{\frac{2^{3/2}-1}{3}}.
\end{aligned}$$

Example 6

Evaluate $\iint_D 2 - x^2 - y^2 \, dA$, where D is the square with vertices $(\pm 1, 0)$ and $(0, \pm 1)$.

Solution. First we sketch the domain D:



This is both Type I and Type II, but no matter how we treat it, the edges will be piecewise defined.

This means our inner integral will have to be broken in half.

However, if we consider the shape of $z = 2 - x^2 - y^2$, we see that the volume computed by the integral is just 4 times the volume in the first octant.

That is

$$\iint_D 2 - x^2 - y^2 \, dA = 4 \int_0^1 \int_0^{1-x} 2 - x^2 - y^2 \, dy \, dx$$

$$= 4 \int_0^1 2y - x^2 y - \frac{y^3}{3} \Bigg|_{y=0}^{y=1-x} dx$$

$$= 4 \int_0^1 2(1-x) - x^2(1-x) - \frac{(1-x)^3}{3} \, dx$$

$$= 4 \int_0^1 \frac{4}{3}x^3 - 2x^2 - x + \frac{5}{3} \, dx$$

$$= 4\left(\frac{1}{3}x^4 - \frac{2}{3}x^3 - \frac{1}{2}x^2 + \frac{5}{3}x\right)\Big|_0^1 = \boxed{\frac{10}{3}}.$$

$\square$

Remark. If we had decided to treat the entire region as Type I, say, we would have gotten the iterated integral

$$\int_{-1}^{1}\int_{|x|-1}^{1-|x|} 2 - x^2 - y^2\, dy\, dx = \int_{-1}^{1} \frac{8}{3}\, x^2\, |x| - 4\, x^2 - 2\, |x| + \frac{10}{3}\, dx$$

$$= 2\int_0^1 \frac{8}{3}x^3 - 4x^2 - 2x + \frac{10}{3}\, dx = 2\left(\frac{2}{3}x^4 - \frac{4}{3}x^3 - x^2 + \frac{10}{3}x\right)\Big|_0^1$$

$$= 2\left(\frac{2}{3} - \frac{4}{3} - 1 + \frac{10}{3}\right) = \frac{10}{3},$$

as expected. Here we have used the fact that $\frac{8}{3}\, x^2\, |x| - 4\, x^2 - 2\, |x| + \frac{10}{3}$ is an even function.