Integration in Polar Coordinates

Ryan C. Daileda

Trinity University

Calculus III

Introduction

Today we will formally introduce polar coordinates.

After giving the fundamental relationships between rectangular and polar coordinates, our first task will be to determine how to express a double integral using polar coordinates.

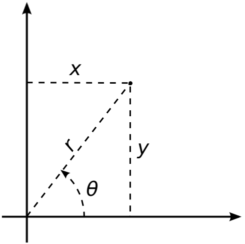
Converting a double integral to polar coordinates can be viewed as a 2D version of u-substitution.

At the end of the lecture we will use polar coordinates to help us find the area underneath the "bell curve" $y = e^{-x^2}$.

Polar Coordinates

Let $P \in \mathbb{R}^2$. The *rectangular coordinates* (x, y) of P describe its position relative to the coordinate axes.

Let θ denote the (counterclockwise) angle between the x-axis and the line segment connecting O to P, and let r be the distance from P to O.



$$x = r\cos\theta,$$

$$y = r\sin\theta,$$

$$x^2 + y^2 = r^2.$$

We can also describe the position of P using (r, θ), and we call these the *polar coordinates* of P.

Remarks

Note that $r \geq 0$, and $r = 0$ if and only if $P = O$.

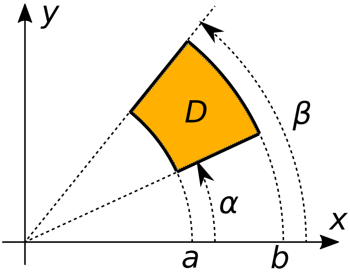
The equation $r = k$ represents a circle around the origin of radius k.

The inequality $a \leq r \leq b$ represents an *annulus* centered at O with radii a and b.

The equation $\theta = k$ represents a ray from O.

The inequality $\alpha \leq \theta \leq \beta$ represents an infinite sector centered at O.

The region D described by the polar inequalities $a \leq r \leq d$ and $\alpha \leq \theta \leq \beta$ is a sector of an annulus (a *polar rectangle*):



Note that the area of D is

$$\pi(b^2 - a^2)\frac{\beta - \alpha}{2\pi} = \frac{b + a}{2}(b - a)(\beta - \alpha).$$

Suppose we are given a function $f(x, y)$ on D. We seek to express $\iint_D f(x, y)\, dA$ in polar coordinates.

Subdivisions

$$a = r_0 < r_1 < r_2 < \cdots < r_m = b,$$
$$\alpha = \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_n = \beta,$$

in r and θ cut D into mn polar subrectangles

$$r_{i-1} \le r \le r_i, \ \ \theta_{j-1} \le \theta \le \theta_j,$$

each with area

$$\frac{r_i + r_{i-1}}{2}(r_i - r_{i-1})(\theta_j - \theta_{j-1}) = \frac{r_i + r_{i-1}}{2} \Delta r_i \, \Delta \theta_j = \Delta A_{ij}.$$

Let P_{ij} be the center of the ij subdivision.

Then P_{ij} has polar coordinates $(r_i^*, \theta_j^*) = \left(\frac{r_i + r_{i-1}}{2}, \frac{\theta_j + \theta_{j-1}}{2}\right)$.

The Riemann sum corresponding to this subdivision with sample points P_{ij} is

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij})\Delta A_{ij} = \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^* \cos\theta_i^*, r_i^* \sin\theta_i^*)\frac{r_i + r_{i-1}}{2}\Delta r_i \, \Delta\theta_j$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^* \cos\theta_i^*, r_i^* \sin\theta_i^*) r_i^* \Delta r_i \, \Delta\theta_j$$

As $\Delta r, \Delta\theta \to 0$, the LHS converges to

$$\iint_D f(x,y)\,dA,$$

while the RHS converges to

$$\int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta.$$

Thus:

Theorem 1 (Integration in Polar Coordinates)

If D denotes the polar rectangle $a < r < b$, $\alpha < \theta < \beta$, and if $f(x,y)$ is integrable on D, then

$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta.$$

Remarks

- Theorem 1 states that if we make the change of variables (substitution)

$$x = r\cos\theta, \quad y = r\sin\theta,$$

then we can formally substitute in the integral, provided we take

$$dA = dx\,dy = r\,dr\,d\theta.$$

- The extra r appearing in the differential is called the *Jacobian* of the polar coordinate transformation.
- Forgetting to include the Jacobian when converting an integral to polar coordinates will almost certainly ruin one's day.

Examples

Example 1

Evaluate $\iint_D xy\,dA$, where D is the portion of $x^2+y^2 \le 9$ in the first quadrant.

Solution. The region D is a quarter of a disk, which is given by the polar inequalities

$$0 \le r \le 3, \;\; 0 \le \theta \le \frac{\pi}{2}.$$

This is a polar rectangle, so we have

$$\iint_D xy\,dA = \int_0^{\pi/2}\int_0^3 (r\cos\theta)(r\sin\theta)r\,dr\,d\theta$$

Because the limits are constants and the integrand is separable, we have

$$\int_0^{\pi/2} \sin\theta \cos\theta \, d\theta \times \int_0^3 r^3 \, dr = \left(\frac{\sin^2\theta}{2} \bigg|_0^{\pi/2} \right) \left(\frac{r^4}{4} \bigg|_0^3 \right)$$

$$= \frac{1}{2} \cdot \frac{81}{4} = \boxed{\frac{81}{8}}.$$

□

Example 2

Evaluate $\iint_D \cos(x^2 + y^2) \, dA$, where D is the portion of the annulus $1 \le x^2 + y^2 \le 4$ that lies above the x-axis.

Remark. This would be extremely difficult to evaluate in rectangular coordinates (why?).

Solution. The annulus is described in polar coordinates by $1 \le r \le 2$ and $0 \le \theta \le \pi$, which is a polar rectangle.

Therefore

$$\begin{aligned}\iint_D \cos(x^2+y^2)\,dA &= \int_0^\pi \int_1^2 \cos(r^2)\,r\,dr\,d\theta \\ &= \int_0^\pi d\theta \times \int_1^2 r\cos(r^2)\,dr \\ &= \pi\left(\left.\frac{\sin(r^2)}{2}\right|_1^2\right) \\ &= \pi\left(\frac{\sin(4)-\sin(1)}{2}\right) = \boxed{\frac{\pi}{2}(\sin(4)-\sin(1))}.\end{aligned}$$

□

Example 3

Evaluate $\displaystyle\iint_D \frac{y^2}{x^2}\,dA$, where
$D = \{(x,y) \mid 4 \le x^2 + y^2 \le 9, |y| \le x\}$.

Solution. The region D is an annular sector with $2 \le r \le 3$ and $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$.
Thus

$$\iint_D \frac{y^2}{x^2}\,dA = \int_{-\pi/4}^{\pi/4}\int_2^3 \frac{r^2\sin^2\theta}{r^2\cos^2\theta}\,r\,dr\,d\theta$$

$$= \int_{-\pi/4}^{\pi/4} \frac{\sin^2\theta}{\cos^2\theta}\,d\theta \times \int_2^3 r\,dr$$

$$= \int_{-\pi/4}^{\pi/4} \frac{1-\cos^2\theta}{\cos^2\theta}\,d\theta \times \left(\frac{r^2}{2}\bigg|_2^3\right)$$

$$= \frac{5}{2}\int_{-\pi/4}^{\pi/4} \sec^2\theta - 1\,d\theta = \frac{5}{2}\left(\tan\theta - \theta\Big|_{-\pi/4}^{\pi/4}\right)$$

$$= \frac{5}{2}\left(1 - \frac{\pi}{4} - \left(-1 + \frac{\pi}{4}\right)\right)$$

$$= \frac{5}{2}\left(2 - \frac{\pi}{2}\right) = \boxed{\frac{5(4-\pi)}{4}}.$$

□

Example 4

Evaluate $\iint_D y\,dA$, where D is the region between the circles $x^2 + y^2 = 4$ and $(x-1)^2 + y^2 = 1$ in the first quadrant.

Solution. We first express both circles in polar coordinates. The larger circle is just $r = 2$.

For the smaller we have

$$\begin{aligned}(x-1)^2+y^2=1 \quad &\Rightarrow \quad x^2-2x+1+y^2=1\\ &\Rightarrow \quad x^2+y^2=2x\\ &\Rightarrow r^2=2r\cos\theta\\ &\Rightarrow r=2\cos\theta.\end{aligned}$$

Radial cross sections begin at $r=2\cos\theta$ and end at $r=2$, as θ varies from 0 to $\pi/2$.

So this is a polar Type I region:

$$\begin{aligned}\iint_D y\,dA &= \int_0^{\pi/2}\int_{2\cos\theta}^{2} r\sin\theta\, r\,dr\,d\theta\\ &= \int_0^{\pi/2} \left.\frac{r^3\sin\theta}{3}\right|_{r=2\cos\theta}^{r=2} d\theta\end{aligned}$$

$$= \frac{8}{3}\int_0^{\pi/2} \sin\theta - \sin\theta\cos^3\theta\, d\theta = \frac{8}{3}\int_0^{\pi/2} (1-\cos^3\theta)\sin\theta\, d\theta$$

$$\begin{pmatrix} u = \cos\theta \\ du = -\sin\theta\, d\theta \end{pmatrix} = \frac{8}{3}\int_1^0 1 - u^3(-du) = \frac{8}{3}\int_0^1 1 - u^3\, du$$

$$= \frac{8}{3}\left(u - \frac{u^4}{4}\bigg|_0^1\right) = \boxed{2}.$$

□

Example 5

Compute the area enclosed by one loop of the polar curve $r = \cos 3\theta$.

Solution. If D denotes the region enclosed by the curve, then

$$\text{Area}(D) = \iint_D dA = \int_\alpha^\beta \int_0^{\cos 3\theta} r\,dr\,d\theta,$$

where α and β are the values of θ for which $r = \cos 3\theta = 0$.

We have

$$\begin{aligned} \cos 3\theta = 0 \;&\Leftrightarrow\; 3\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots \\ &\Leftrightarrow\; \theta = \pm\frac{\pi}{6}, \pm\frac{\pi}{2}, \pm\frac{5\pi}{6}, \ldots \end{aligned}$$

The first loop is given by $-\pi/6 \le \theta \le \pi/6$.

Therefore the area is

$$\int_{-\pi/6}^{\pi/6}\int_0^{\cos 3\theta} r\,dr\,d\theta = \int_{-\pi/6}^{\pi/6} \left.\frac{r^2}{2}\right|_{r=0}^{r=\cos 3\theta} d\theta$$

$$= \frac{1}{2}\int_{-\pi/6}^{\pi/6} \cos^2 3\theta\,d\theta$$

$$= \frac{1}{2}\int_{-\pi/6}^{\pi/6} \frac{1+\cos 6\theta}{2}\,d\theta$$

$$= \frac{1}{4}\left(\theta + \left.\frac{\sin 6\theta}{12}\right|_{-\pi/6}^{\pi/6}\right)$$

$$= \frac{1}{4}\cdot\frac{2\pi}{6} = \boxed{\frac{\pi}{12}}.$$

□

Example 6

Find the area underneath the curve $y = e^{-x^2}$ and the x-axis.

Solution. The area is given by the (improper) integral

$$A = \int_{-\infty}^{\infty} e^{-x^2}\,dx = \int_{-\infty}^{\infty} e^{-y^2}\,dy.$$

We cannot evaluate this using FTOC, so we work indirectly. Notice that

$$A^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy$$

By Fubini's theorem, this is equal to

$$\iint_{\mathbb{R}^2} e^{-x^2-y^2}\,dA = \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\,r\,dr\,d\theta$$
$$= 2\pi\int_0^{\infty} re^{-r^2}\,dr$$
$$= -\pi\left(e^{-r^2}\Big|_0^{\infty}\right)$$
$$= -\pi(0-1) = \pi.$$

Thus

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = A = \boxed{\sqrt{\pi}}.$$