Change of Variables in Double Integrals

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Calculus III

Introduction

Consider the integral $\int_a^b f(x)\,dx$. The substitution $x = g(t)$ has two effects on the integral:

- It replaces the differential dx with $g'(t)\,dt$.
- It replaces the interval $[a, b]$ with $[g^{-1}(a), g^{-1}(b)]$.

We will see that a *change of variables* $x = x(u, v)$, $y = y(u, v)$ has a similar effect on $\iint_R f(x, y)\,dA$:

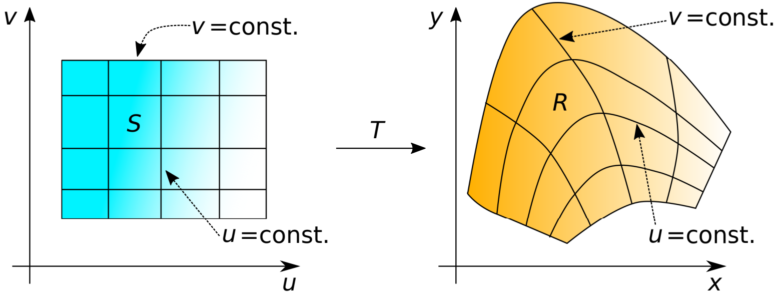
- It replaces the differential dA with $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\,du\,dv$, where $\frac{\partial(x,y)}{\partial(u,v)}$ is the so-called *Jacobian*.
- It replaces the region R in the xy-plane with a certain region S in the uv-plane.

Coordinate Transformations

Let $R \subset \mathbb{R}^2$, considered in the xy-plane.

Let $S \subset \mathbb{R}^2$, considered in the uv-plane.

A *(two-dimensional) coordinate transformation* is a bijective function $T : S \to R$.



If we write $T(u,v) = (x(u,v), y(u,v))$, we call $x(u,v)$ and $y(u,v)$ the *coordinate functions* of T.

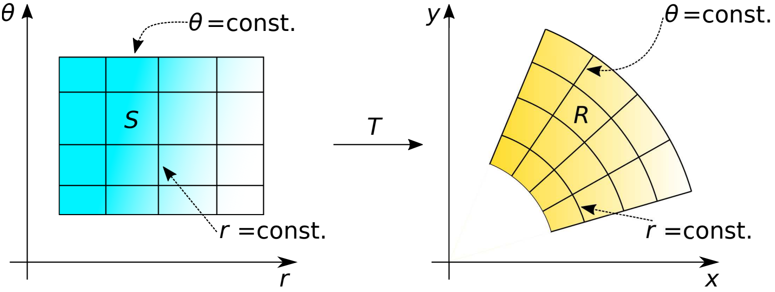
The map T represents a *change of variables* from the uv- to the xy-coordinate system.

We will say that T is *smooth* if $x(u,v)$ and $y(u,v)$ have continuous first order derivatives.

Example

The *polar coordinate transformation* is given by

$$(x, y) = T(r, \theta) = (r\cos\theta, r\sin\theta).$$



Given a smooth coordinate transformation $T : S \to R$, any function $f(x, y)$ on R becomes a function $F(u, v)$ on S:

$$F(u, v) = f(x(u, v), y(u, v)).$$

We would like to relate the integral of f on R to an integral involving F on S.

Let $[u_0, u_0 + \Delta u] \times [v_0, v_0 + \Delta v]$ be a small subrectangle of S.

Because T is smooth, the coordinate functions $x(u, v)$ and $y(u, v)$ are differentiable.

This means that if Δu and Δv are small enough,

$$x(u,v) \approx x(u_0,v_0) + \frac{\partial x}{\partial u}(u-u_0) + \frac{\partial x}{\partial v}(v-v_0) = \widehat{x}(u,v),$$
$$y(u,v) \approx y(u_0,v_0) + \frac{\partial y}{\partial u}(u-u_0) + \frac{\partial y}{\partial v}(v-v_0) = \widehat{y}(u,v),$$

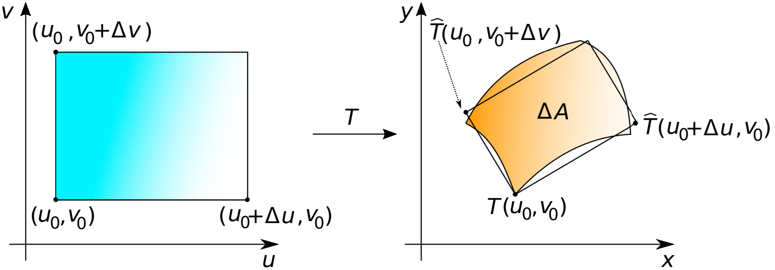
for $(u,v) \in [u_0, u_0+\Delta u] \times [v_0, v_0+\Delta v]$.

The *linearization* $\widehat{T}(u,v) = (\widehat{x}(u,v), \widehat{y}(u,v))$ carries rectangles to parallelograms, and maps vertices to corresponding vertices.

$\widehat{T}$ therefore carries $[u_0, u_0 + \Delta u] \times [v_0, v_0 + \Delta v]$ to the parallelogram with vertex $(x_0, y_0) = (u_0, v_0)$ and edges

$$\widehat{T}(u_0 + \Delta u, v_0) - \widehat{T}(u_0, v_0) = \left\langle \frac{\partial x}{\partial u}\Delta u, \frac{\partial y}{\partial u}\Delta u \right\rangle = \Delta u \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u} \right\rangle,$$

$$\widehat{T}(u_0, v_0 + \Delta v) - \widehat{T}(u_0, v_0) = \left\langle \frac{\partial x}{\partial v}\Delta v, \frac{\partial y}{\partial v}\Delta v \right\rangle = \Delta v \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v} \right\rangle.$$



So, for small Δu, Δv:

$$\begin{aligned}\Delta A &= \text{Area}\left(T([u_0, u_0 + \Delta u] \times [v_0, v_0 + \Delta v])\right) \\ &\approx \text{Area}\left(\widehat{T}([u_0, u_0 + \Delta u] \times [v_0, v_0 + \Delta v])\right) \\ &= \left|\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}\right| \Delta u \, \Delta v.\end{aligned}$$

The determinant

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

is called the *Jacobian* of T.

Therefore, if we subdivide S, apply T and form the corresponding Riemann sum in the xy-plane, we get

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A \approx \sum_{i=1}^{m}\sum_{j=1}^{m} f(x(u_{ij}^*, v_{ij}^*), y(u_{ij}^*, v_{ij}^*)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \Delta u\,\Delta v.$$

As $\Delta u, \Delta v \to 0$, the approximation improves indefinitely, and we get the equality

$$\boxed{\iint_R f(x,y)\,dx\,dy = \iint_S f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv}.$$

Example

The polar coordinate transformation $T(r,\theta) = (r\cos\theta, r\sin\theta)$ carries the rectangle $[a,b] \times [\alpha,\beta]$ to an annular sector R in the xy-plane.

The Jacobian is given by

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

We therefore arrive at the polar coordinate change of variables formula

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta.$$

Example

Example 1

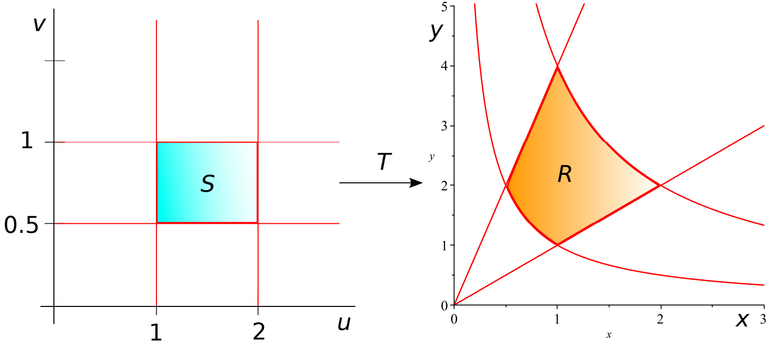
Evaluate $\iint_R e^{xy}\,dA$, where R is the region in the first quadrant bounded by $xy = 1$, $xy = 4$, $y = x$ and $y = 4x$.

Solution. The transformation $T(u, v) = (uv, u/v)$ carries the line $u = k$ to the points $T(k, v) = (kv, k/v) = (x, y)$, which satisfy $xy = k^2$.

T also carries the line $v = k$ to the points $T(u, k) = (ku, u/k)$ $= (x, y)$, which satisfy $x/y = k^2$.

That is, T carries vertical lines to hyperbolas, and horizontal lines to lines through the origin.

The region R in the first quadrant bounded by $xy = 1$, $xy = 4$, $y = x$ and $y = 4x$ is thus the T-image of the rectangle $S = [1,2] \times [1/2, 1]$ in the uv-plane.



The Jacobian of T is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} v & u \\ 1/v & -u/v^2 \end{vmatrix} = -\frac{2u}{v}.$$

Thus,

$$\begin{aligned}
\iint_R e^{xy}\,dA &= \int_{1/2}^{1}\int_1^2 e^{(uv)(u/v)}\,\frac{2u}{v}\,du\,dv \\
&= \int_{1/2}^{1}\frac{dv}{v}\int_1^2 2ue^{u^2}\,du \\
&= \left(\ln v\Big|_{1/2}^{1}\right)\left(e^{u^2}\Big|_1^2\right) \\
&= \boxed{\ln 2\left(e^4 - e\right)}.
\end{aligned}$$

□

Example 2

Evaluate $\iint_R 2 - x^2 - y^2\, dA$, where R is the square with vertices $(\pm 1, 0)$, $(0, \pm 1)$.

Solution. The (linear) transformation

$$T(u,v) = ((u+v)/2, (u-v)/2)$$

carries the square $S = [-1,1] \times [-1,1]$ bijectively onto R.

Its Jacobian is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{1}{2}.$$

Thus

$$\iint_R 2 - x^2 - y^2\, dA = \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1} 2 - \left(\frac{u+v}{2}\right)^2 - \left(\frac{u-v}{2}\right)^2 du\, dv$$

$$= \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1} 2 - \frac{u^2}{2} - \frac{v^2}{2}\, du\, dv$$

$$= \frac{1}{2}\int_{-1}^{1}\left(2u - \frac{u^3}{6} - \frac{uv^2}{2}\Big|_{u=-1}^{u=1}\right) dv$$

$$= \frac{1}{2}\int_{-1}^{1}\frac{11}{3} - v^2\, dv = \frac{1}{2}\left(\frac{11}{3}v - \frac{v^3}{3}\Big|_{-1}^{1}\right)$$

$$= \frac{1}{2}\cdot\frac{20}{3} = \boxed{\frac{10}{3}},$$

as computed earlier. □

Example 3

Evaluate $\iint_R x^2\,dA$, where R is the region bounded by the ellipse $9x^2 + 4y^2 = 36$.

Solution. Let $(x, y) = T(u, v) = (2u, 3v)$. Then

$$9x^2 + 4y^2 = 36 \iff 9(2u)^2 + 4(3v)^2 = 36 \iff u^2 + v^2 = 1.$$

Therefore T carries the unit disk D in the uv-plane onto R.

The Jacobian of T is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6.$$

Thus,

$$\iint_R x^2 \, dA = \iint_D 4u^2 \cdot 6 \, du \, dv.$$

The integral over D is most easily computed in polar coordinates:

$$\begin{aligned} 24 \iint_D u^2 \, du \, dv &= 24 \int_0^{2\pi} \int_0^1 r^2 \cos^2\theta \, r \, dr \, d\theta \\ &= 8 \int_0^{2\pi} \frac{1 + \cos 2\theta}{2} \, d\theta \\ &= 4 \left(\theta + \frac{\sin 2\theta}{2} \Bigg|_0^{2\pi} \right) = \boxed{8\pi}. \end{aligned}$$

□