# Points in $\mathbb{R}^2$ and $\mathbb{R}^3$

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Calculus III

# Introduction

In Calculus I and II one studies "smooth" (continuous, differentiable, etc.) functions of a single (real) variable.

In Calculus III we will study (smooth) functions of several (real) variables.

We may regard the inputs/outputs of such functions as points in multidimensional Euclidean space.

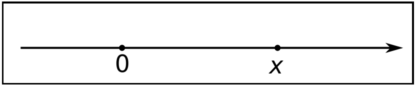
We will therefore begin our course with a discussion of the Euclidean space $\mathbb{R}^n$.

# Definitions and Notation

$\mathbb{R}$: One-dimensional (Euclidean) space

- **Analytically:** the set of all real numbers
- **Geometrically:** the real line

$\boxed{x \in \mathbb{R}}$ $\longleftrightarrow$

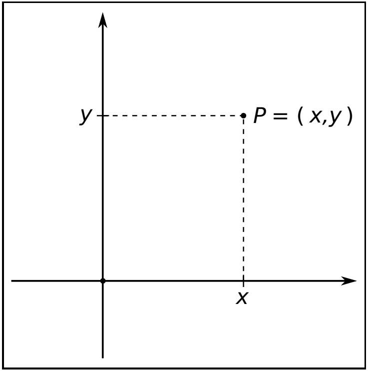



The numerical value of $x$ gives its position on the line (relative to 0).

## $\mathbb{R}^2$: Two-dimensional (Euclidean) space

- **Analytically:** $\{(x, y) \mid x, y \in \mathbb{R}\}$, the set of all *ordered pairs* of real numbers
- **Geometrically:** points in a plane

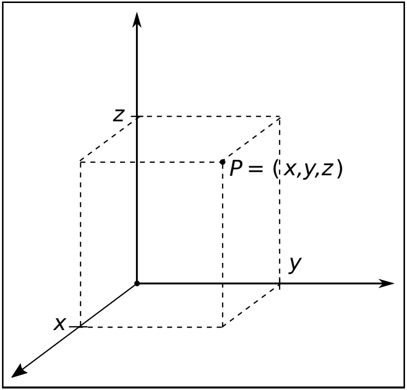
$$\boxed{(x, y) \in \mathbb{R}^2} \quad \longleftrightarrow$$



## $\underline{\mathbb{R}^3:}$ Three-dimensional (Euclidean) space

- **Analytically:** $\{(x, y, z) \mid x, y, z \in \mathbb{R}\}$, the set of all *ordered triples* of real numbers
- **Geometrically:** points in space

$$\boxed{(x, y, z) \in \mathbb{R}^3} \quad \longleftrightarrow$$



# Terminology of $\mathbb{R}^3$

- **(Coordinate) Axes:** The $x$, $y$ and $z$-axes.
- **Origin:** Where the axes intersect, $(0,0,0)$.
- **Coordinate Planes:** Planes formed by pairs of axes. Called the $xy$, $yz$ and $xz$-planes.
- **Octants:** The 8 regions of space cut off by the coordinate planes.

# Remarks

- The arrangement of axes we are using is called *right-handed*.
- Note that the $x$ and $y$-axes have the same relative orientation (when viewed from "above") as they do in $\mathbb{R}^2$.
- We will therefore frequently identify $\mathbb{R}^2$ with the $xy$-plane in $\mathbb{R}^3$.
- The octants have a standard numbering that we will almost never use. The first octant is shown in the diagram above.
- Negative coordinates occur in the various other octants, "behind" or "below" the coordinate planes.

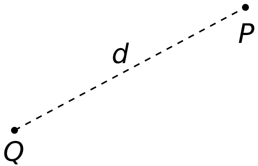
# Generalized Euclidean Space

$\underline{\mathbb{R}^n:}$ $n$-dimensional (Euclidean) space

- **Analytically:** $\{(x_1, x_2, \ldots, x_n) \,|\, x_i \in \mathbb{R} \text{ for all } i\}$, the set of all *n-tuples* of real numbers
- **Geometrically:** difficult to visualize
- When $n$ is small, we usually avoid subscripts and just use extra variables, e.g. $(x, y, z, t) \in \mathbb{R}^4$.
- We will primarily be interested in $n = 2, 3$, but may encounter higher dimensions on occasion.

# Distances

The geometric concept of distance in Euclidean spaces has a straightforward analytic description.




<u>$\mathbb{R}^2$:</u> If $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are points in $\mathbb{R}^2$, the distance between them is

$$d(P, Q) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

$\underline{\mathbb{R}^3:}$ If $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ are points in $\mathbb{R}^3$, the distance between them is

$$d(P, Q) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

**Remarks.**

- These formulae follow easily from the Pythagorean theorem.
- Distance in $\mathbb{R}$ fits this pattern as well. If $x_1, x_2 \in \mathbb{R}$, then

$$d(x_1, x_2) = |x_1 - x_2| = \sqrt{(x_1 - x_2)^2}.$$

# Distance in General

By analogy, the distance from $P = (x_1, x_2, \ldots, x_n)$ to $Q = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$ is *defined* to be

$$d(P, Q) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$$

If $n \geq 4$, then $n$-dimensional distance cannot be visualized.

Nonetheless it obeys the "standard" properties of distance, such as the *triangle inequality,*

$$d(P, Q) \leq d(P, R) + d(R, Q).$$

# More Remarks

- Distances are not always "simple" numbers and frequently involve radicals.

- For example, the distance from $P = (1, 2, 3)$ to $Q = (-3, 1, 0)$ is

$$d(P, Q) = \sqrt{(1 - (-3))^2 + (2 - 1)^2 + (3 - 0)^2}$$
$$= \sqrt{16 + 1 + 9} = \boxed{\sqrt{26}}.$$

- Although a calculator reports that $\sqrt{26} = 5.0990$ (to four decimal places), the *exact* distance is the symbolic expression $\sqrt{26}$. The decimal expression 5.0990 is only an *approximation*.

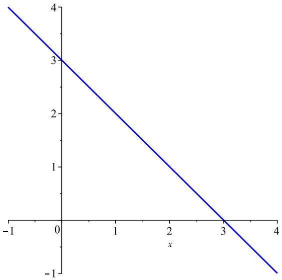
# Equations and Graphs

Roughly speaking we know that:

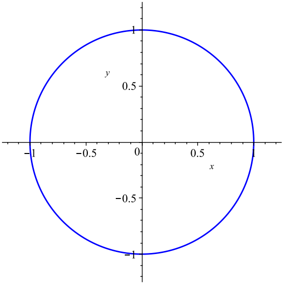
| "Nice" equations in $x, y$ | $\longleftrightarrow$ | Curves in $\mathbb{R}^2$ (the $xy$-plane) |
| --- | --- | --- |
| (Equation $E$) | $\longleftrightarrow$ | (Points that make $E$ true) |

**Examples:**

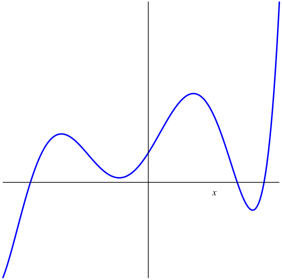
$x + y = 3 \quad \longleftrightarrow$



$x^2 + y^2 = 1 \quad \longleftrightarrow$




$y = f(x) \quad \longleftrightarrow$



# Equations in 3 Variables

In the same way we have a three-dimensional correspondence

$$\boxed{\text{“Nice” equations in } x, y, z} \longleftrightarrow \boxed{\text{Surfaces in } \mathbb{R}^3 \ (xyz\text{-space})}$$

**Examples:** Consider the following equations in $x, y, z$ and describe their corresponding surfaces in $\mathbb{R}^3$.

- $z = 0 \longleftrightarrow xy$-plane
- $y = 0 \longleftrightarrow xz$-plane
- $x = 0 \longleftrightarrow yz$-plane
- $z = -1 \longleftrightarrow xy$-plane shifted “down” one unit

- $x^2 + y^2 = 1 \longleftrightarrow$ unit cylinder around $z$-axis
- $d(P,(x,y,z)) = r \longleftrightarrow$ sphere of radius $r$, centered at $P = (a, b, c)$

We can make the last example more explicit using the distance formula:

$$d(P,(x,y,z)) = r \Leftrightarrow \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} = r$$
$$\Leftrightarrow \underbrace{(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2}_{\text{sphere, radius } r\text{, center } (a,b,c)}$$

# Examples

**Example:** Find an equation for the sphere centered at $P = (1, 2, 3)$ that passes through $Q = (3, 5, -3)$.

**Solution:** Because $Q$ is on the sphere, the radius must be

$$r = d(P, Q) = \sqrt{(1-3)^2 + (2-5)^2 + (3-(-3))^2}$$
$$= \sqrt{49} = 7.$$

Since $P$ is the center, the equation must therefore be

$$\boxed{(x-1)^2 + (y-2)^2 + (z-3)^2 = 49}.$$

**Example:** Determine the surface represented by the equation

$$x^2 + y^2 + z^2 + 4x - 2y + 2z + 1 = 0.$$

**Solution:** We group terms with common variables and complete the squares:

$$\begin{aligned} (x^2 + 4x) + (y^2 - 2y) + (z^2 + 2z) + 1 &= 0 \\ (x + 2)^2 - 4 + (y - 1)^2 - 1 + (z + 1)^2 - 1 + 1 &= 0 \end{aligned}$$

Now collect all the constants on the RHS:

$$(x + 2)^2 + (y - 1)^2 + (z + 1)^2 = 5.$$

This is an equation for the sphere centered at $(-2, 1, -1)$ with radius $\sqrt{5}$.