Triple Integrals

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Calculus III

Introduction

We will now extend out theory of integration to functions of 3 variables.

The simplest generalization of our work so far is to integrate over solid regions (3-manifolds) in $\mathbb{R}^3$.

As before, we will begin with the simplest 3D solids, and then move on to more general regions.

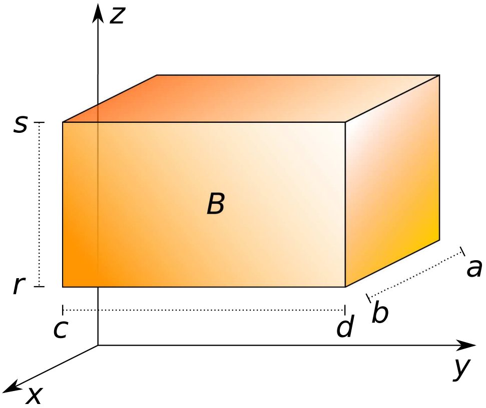
Triple integrals can be difficult to work with because they often require careful 3D visualization.

Triple Integrals over Rectangular Solids

Suppose we are given a function $f(x, y, z)$ with domain

$$B = [a, b] \times [c, d] \times [r, s],$$

a rectangular solid (a "box").

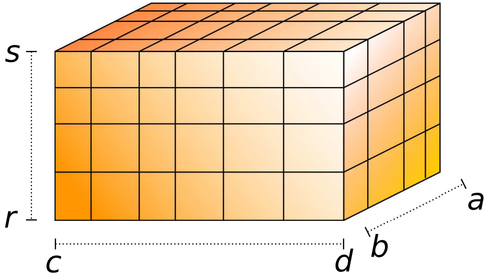


We introduce subdivisions

$$x_0 = a < x_1 < x_2 < \cdots < x_{\ell-1} < x_\ell = b,$$
$$y_0 = c < y_1 < y_2 < \cdots < y_{m-1} < y_m = d,$$
$$z_0 = r < z_1 < z_2 < \cdots < z_{n-1} < z_n = s,$$

and obtain rectangular subdivisions B_{ijk} of B:

$$B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k].$$



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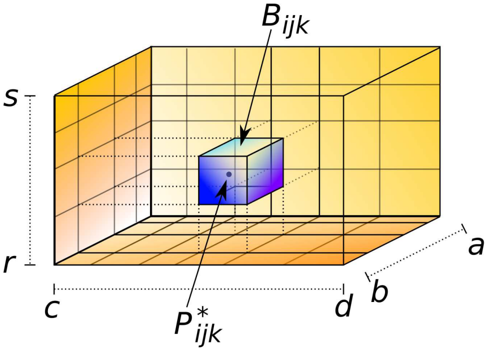


We choose sample points $P^*_{ijk} \in B_{ijk}$ and form the Riemann sum

$$\sum_{i=1}^{\ell} \sum_{j=1}^{m} \sum_{k=1}^{n} f(P^*_{ijk}) \Delta V_{ijk},$$

where

$$\Delta V_{ijk} = \text{ Volume of } B_{ijk} = \Delta x_i \Delta y_j \Delta z_k.$$

Finally we take the limit as the subdivisions shrink indefinitely:

$$\lim_{\Delta x, \Delta y, \Delta z \to 0} \sum_{i=1}^{\ell} \sum_{j=1}^{m} \sum_{k=1}^{n} f(P^*_{ijk}) \Delta V_{ijk} = \iiint_B f(x, y, z)\, dV.$$

We say that f is *integrable* when this limit exists (and is independent of the choices involved).

Interpreting the Triple Integral

Unfortunately $\iiint_B f\,dV$ does *not* have a "generic" geometric meaning.

The terms $f(P_{ijk}^*)\Delta V_{ijk}$ in the Riemann sum represent 4-dimensional "content," which cannot be visualized.

However, if f represents the pointwise density throughout B, then:

- $f(P_{ijk}^*)\Delta V_{ijk}$ is the approximate mass of B_{ijk}.
- $\iiint_B f\,dV$ is the total mass of B.

If the subdivisions of B are uniform, then

$$\Delta V_{ijk} = \Delta V = \frac{\text{Volume of B}}{\ell mn} \quad \Rightarrow \quad \frac{1}{\ell mn} = \frac{\Delta V}{V(B)},$$

and the average value of f at the sample points P_{ijk}^* is

$$\frac{1}{\ell mn} \sum_{i=1}^{\ell} \sum_{j=1}^{m} \sum_{k=1}^{n} f(P_{ijk}^*) = \frac{1}{V(B)} \sum_{i=1}^{\ell} \sum_{j=1}^{m} \sum_{k=1}^{n} f(P_{ijk}^*) \, \Delta V.$$

Taking the limit as $\Delta x, \Delta y, \Delta z \to 0$ we sample f "everywhere" and find

$$\boxed{\text{Average value of } f \text{ on } B = \overline{f} = \frac{1}{V(B)} \iiint_B f \, dV}.$$

This provides another interpretation of the triple integral.

Fubini's Theorem

As with double integrals, we can use a cross section argument to show that triple integrals can be evaluated using iterated single variable integrals.

Theorem 1

If f is integrable on $B = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint_B f(x, y, z)\, dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz$$
$$= \int_a^b \int_r^s \int_c^d f(x, y, z)\, dy\, dz\, dx = \cdots$$

(There are six possible orders.)

Example 1

Evaluate $\iiint_B xz - y^3 \, dV$, where $B = [-1,1] \times [0,2] \times [0,1]$.

Solution. There is no particularly advantageous order of integration, so we simply choose

$$\iiint_B xz - y^3 \, dV = \int_{-1}^{1} \int_0^2 \int_0^1 xz - y^3 \, dz \, dy \, dx$$

$$= \int_{-1}^{1} \int_0^2 \frac{xz^2}{2} - y^3 z \Big|_{z=0}^{z=1} dy \, dx$$

$$= \int_{-1}^{1} \int_0^2 \frac{x}{2} - y^3 \, dy \, dx = \int_{-1}^{1} \frac{xy}{2} - \frac{y^4}{4} \Big|_{y=0}^{y=2} dx$$

$$= \int_{-1}^{1} x - 4\,dx = \frac{x^2}{2} - 4x\Big|_{-1}^{1} = \boxed{-8}.$$

□

Remarks.

- This result tells us that $xz - y^3$ is predominantly negative throughout B.
- More precisely, the average value of $xz - y^3$ in B is

$$\frac{1}{V(B)} \iiint_B xz - y^3\,dV = \frac{1}{2 \cdot 2 \cdot 1}(-8) = -2.$$

Triple Integrals Over General Regions

To integrate over a general solid region, we first divide it into subregions that can be described by iterated inequalities.

We can use a cross section argument to express these subintegrals as 3-variable iterated integrals.

We then appeal to the *additivity* of the integral: if E_1 and E_2 are solids only sharing boundary points, then

$$\iiint_{E_1 \cup E_2} f \, dV = \iiint_{E_1} f \, dV + \iiint_{E_2} f \, dV.$$

Definition

We say that $E \subset \mathbb{R}^3$ is of *Type 1* if it has the form

$$E = \{(x, y, z) \,|(x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\},$$

where $D \subset \mathbb{R}^2$ is the projection of E into the xy-plane. In this case

$$\iiint_E f(x, y, z)\, dV = \iint_D \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz\, dA,$$

where $dA = dx\, dy = dy\, dx$.

A Type 1 region has a "bottom" surface given by $z = u_1(x, y)$, and a "top" surface given by $z = u_2(x, y)$, both with domain D in the xy-plane (see Maple diagram).

Definition

We say that $E \subset \mathbb{R}^3$ is of *Type 2* if it has the form

$$E = \{(x,y,z) \,|(y,z) \in D, u_1(y,z) \le x \le u_2(y,z)\},$$

where $D \subset \mathbb{R}^2$ is the projection of E into the yz-plane. In this case

$$\iiint_E f(x,y,z)\,dV = \iint_D \int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx\,dA,$$

where $dA = dy\,dz = dz\,dy$.

A Type 2 region has a "back" surface given by $x = u_1(y,z)$, and a "front" surface given by $x = u_2(y,z)$, both with domain D in the yz-plane (see Maple diagram).

Definition

We say that $E \subset \mathbb{R}^3$ is of *Type 3* if it has the form

$$E = \{(x, y, z) \,|(x, z) \in D, u_1(x, z) \leq y \leq u_2(x, z)\},$$

where $D \subset \mathbb{R}^2$ is the projection of E into the xz-plane. In this case

$$\iiint_E f(x, y, z)\, dV = \iint_D \int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z)\, dy\, dA,$$

where $dA = dx\, dz = dz\, dx$.

A Type 3 region has a "left" surface given by $y = u_1(x, z)$, and a "right" surface given by $y = u_2(x, z)$, both with domain D in the xz-plane (see Maple diagram).

Remarks

Fubini's Theorem still holds for general regions: if E is of multiple types, $\iiint_E f\,dV$ can be computed using any order of integration.

A triple integral $\iiint_E f\,dV$ can be very difficult to set up.

- One first needs to identify the type of E and the relevant "sides" to determine the innermost variable and limits of integration.
- Then one needs to project E into the plane of the remaining variables and set up a double integral over that region.

When $f \equiv 1$, the terms in the Riemann sums are simply volumes, so that $\iiint_E dV =$ Volume of E.

Examples

Example 2

Compute $\iiint_E z\,dV$ where E is the solid region bounded by $z = 1 - x^2 - y^2$ and $z = -3$.

Solution. The paraboloid $z = 1 - x^2 - y^2$ lies above $z = -3$ where

$$1 - x^2 - y^2 \geq -3 \;\Leftrightarrow\; x^2 + y^2 \leq 4.$$

So E is a Type 1 region with "bottom" $z = -3$, "top" $z = 1 - x^2 - y^2$ and xy-projection

$$D = \{(x, y) \,|\, x^2 + y^2 \leq 4\}.$$

See Maple diagram.

Thus

$$\iiint_E z\,dV = \iint_D \int_{-3}^{1-x^2-y^2} z\,dz\,dA.$$

Because the outer integral is over a disk centered at the origin, we can easily describe it using polar coordinates:

$$\iint_D \int_{-3}^{1-x^2-y^2} z\,dz\,dA = \int_0^{2\pi}\int_0^2\int_{-3}^{1-r^2} zr\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 \frac{rz^2}{2}\bigg|_{z=-3}^{z=1-r^2} dr\,d\theta = \frac{1}{2}\int_0^{2\pi}\int_0^2 r(1-r^2)^2 - 9r\,dr\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \frac{(1-r^2)^3}{-6} - \frac{9r^2}{2}\bigg|_{r=0}^{r=2} d\theta = \pi\left(\frac{9}{2} - 18 + \frac{1}{6}\right) = \boxed{\frac{-40\pi}{3}}.$$

□

Example 3

Find the volume of the region E enclosed by the parabolic cylinder $z = 3 - y^2$ and the planes $z = 0$, $x + y + z = 5$ and $x - y = -7$.

Solution. The plane $z = 0$ cuts the cylinder $z = 3 - y^2$ into a "parabolic tube" along the x-axis.

The planes $x + y + z = 5$ and $x - y = -7$ intersect the x-axis at $x = 5$ and $x = -7$, respectively.

Therefore $x = 5 - y - z$ forms the "front" of E, while $x = y - 7$ forms the "back."

Thus E is a Type 2 region. See Maple diagram.

The projection of E into the yz-plane is the parabolic region D bounded by $z = 0$ and $z = 3 - y^2$, which is of Type I.

Thus

$$\text{Volume}(E) = \iiint_E dV = \iint_D \int_{y-7}^{5-y-z} dx\, dA$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}} \int_0^{3-y^2} \int_{y-7}^{5-y-z} dx\, dz\, dy = \int_{-\sqrt{3}}^{\sqrt{3}} \int_0^{3-y^2} 12 - 2y - z\, dz\, dy$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}} (12 - 2y)z - \frac{z^2}{2} \bigg|_{z=0}^{z=3-y^2} dy$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}} (12 - 2y)(3 - y^2) - \frac{(3 - y^2)^2}{2}\, dy$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}} \frac{-y^4}{2} + 2y^3 - 9y^2 - 6y + \frac{63}{2}\, dy$$

$$= 2\int_0^{\sqrt{3}} -\frac{y^4}{2} - 9y^2 + \frac{63}{2}\, dy = \int_0^{\sqrt{3}} -y^4 - 18y^2 + 63\, dy$$

$$= \frac{-y^5}{5} - 6y^3 + 63y\Big|_0^{\sqrt{3}} = \frac{-9\sqrt{3}}{5} - 18\sqrt{3} + 63\sqrt{3}$$

$$= \boxed{\frac{216}{5}\sqrt{3}}.$$

□

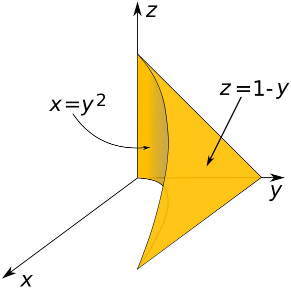
Example 4

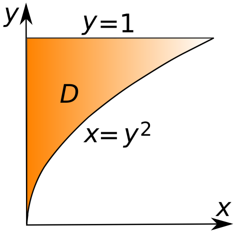
Rewrite the integral $\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z)\,dz\,dy\,dx$ in all five other orders of integration.

Solution. The given iterated integral represents an integral over a Type 1 region $E \subset \mathbb{R}^3$ bounded above by the plane $z = 1 - y$ and below by the plane $z = 0$.

The xy-projection D of E is a Type I region bounded on the bottom by the parabola $y = \sqrt{x}$ $(x = y^2)$ and on the top by the horizontal line $y = 1$, running from $x = 0$ to $x = 1$.

As a Type II region, D is bounded on the left by $x = 0$ and on the right by$x = y^2$, with $0 \leq y \leq 1$.

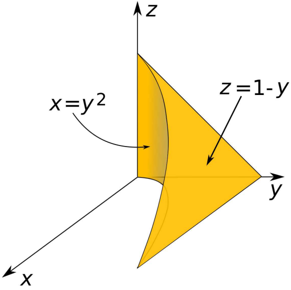


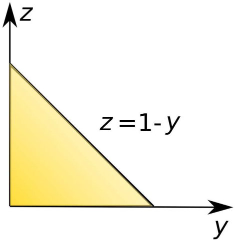


Thus

$$\iiint_E f(x,y,z)\,dV = \boxed{\int_0^1 \int_0^{y^2} \int_0^{1-y} f(x,y,z)\,dz\,dx\,dy}.$$

As a Type 2 region, the "back" of E is $x = 0$, while the "front" is $x = y^2$.



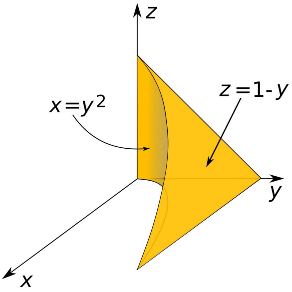


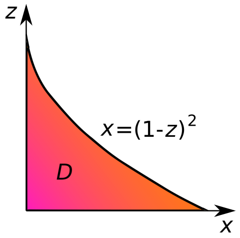
The yz-projection of E is a triangle in the first quadrant. Thus

$$\iiint_E f(x,y,z)\,dV = \boxed{\int_0^1 \int_0^{1-y} \int_0^{y^2} f(x,y,z)\,dx\,dz\,dy}$$

$$= \boxed{\int_0^1 \int_0^{1-z} \int_0^{y^2} f(x,y,z)\,dx\,dy\,dz}.$$

As a Type 3 region, the "left side" of E is $y = \sqrt{x}$, while the "right side" is $y = 1 - z$.





The xz-projection of E is bounded by $x = y^2 = (1 - z)^2$. Thus

$$\iiint_E f(x,y,z)\,dV = \boxed{\int_0^1 \int_0^{1-\sqrt{x}} \int_{\sqrt{x}}^{1-z} f(x,y,z)\,dy\,dz\,dx}$$

$$= \boxed{\int_0^1 \int_0^{(1-z)^2} \int_{\sqrt{x}}^{1-z} f(x,y,z)\,dy\,dx\,dz}.$$