Triple Integrals in Cylindrical and Spherical Coordinates

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Calculus III

Introduction

As with double integrals, it can be useful to introduce other 3D coordinate systems to facilitate the evaluation of triple integrals.

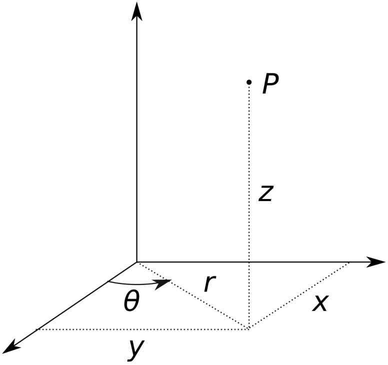
We will primarily be interested in two particularly useful coordinate systems: cylindrical and spherical coordinates.

Cylindrical coordinates are closely connected to polar coordinates, which we have already studied.

Spherical coordinates, however, are a truly “new” coordinate system, so we will spend more time studying them.

Cylindrical Coordinates

Given a point $P \in \mathbb{R}^3$ with rectangular coordinates (x, y, z), its *cylindrical coordinates* (r, θ, z) are defined by:



$$
\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta \\
z &= z \\
r^2 &= x^2 + y^2 \\
dV &= dx\,dy\,dz = r\,dz\,dr\,d\theta
\end{aligned}
$$

That is, we use the usual z-coordinate, but work in polar coordinates in the xy-plane

Examples

Cylindrical coordinates are useful for describing regions $E \subset \mathbb{R}^3$ whose xy-projections have "nice" polar descriptions.

Example 1

Evaluate $\iiint_E e^z \, dV$ where E is enclosed by $z = 1 + x^2 + y^2$, $x^2 + y^2 = 5$ and the xy-plane.

Solution. The region E is Type 1 with "bottom" $z = 0$, "top" $z = 1 + x^2 + y^2$, and xy-projection $x^2 + y^2 \leq 5$.

See Maple diagram.

Using cylindrical coordinates we have

$$\iiint_E e^z\,dV = \int_0^{2\pi}\int_0^{\sqrt{5}}\int_0^{1+r^2} e^z\,r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \times \int_0^{\sqrt{5}}\int_0^{1+r^2} e^z\,r\,dz\,dr$$

$$= 2\pi\int_0^{\sqrt{5}} e^z r\Big|_{z=0}^{z=1+r^2} dr = 2\pi\int_0^{\sqrt{5}} re^{1+r^2} - r\,dr$$

$$= \pi\left(e^{1+r^2} - r^2\Big|_0^{\sqrt{5}}\right) = \boxed{\pi(e^6 - e - 5)}.$$

□

Example 2

Evaluate $\iiint_E x\,dV$ where E is enclosed by $z = 0$, $z = x + y + 5$, $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$.

Solution. The region E is the portion of the tube centered on the z-axis, with inner radius 2, outer radius 3, between the planes $z = 0$ (the "bottom") and $z = x + y + 5$ (the "top").

See Maple diagram.

This is a Type 1 region, with cylindrical description $0 \leq z \leq r\cos\theta + r\sin\theta + 5$, $2 \leq r \leq 3$ and $0 \leq \theta \leq 2\pi$.

Therefore

$$\iiint_E x\,dV = \int_0^{2\pi}\int_2^3\int_0^{r\cos\theta + r\sin\theta+5} r\cos\theta\, r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_2^3 r^2\cos\theta\left(r\cos\theta + r\sin\theta + 5\right)\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_2^3 r^3\cos^2\theta + r^3\sin\theta\cos\theta + 5r^2\cos\theta\,dr\,d\theta$$

$$= \int_2^3\int_0^{2\pi} r^3\left(\frac{1+\cos 2\theta}{2}\right) + r^3\sin\theta\cos\theta + 5r^2\cos\theta\,d\theta\,dr$$

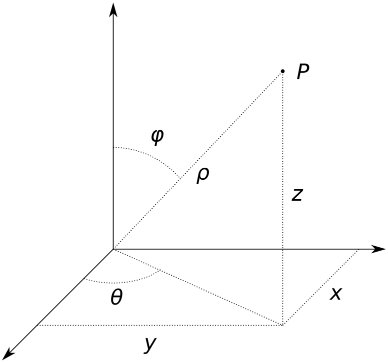
$$= \int_2^3 r^3\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right) + \frac{r^2\sin^2\theta}{2} + 5r^2\sin\theta\Big|_{\theta=0}^{\theta=2\pi}\,dr$$

$$= \pi\int_2^3 r^3\,dr = \pi\frac{r^4}{4}\Big|_2^3 = \boxed{\frac{65\pi}{4}}.$$

□

Spherical Coordinates

Given a point $P \in \mathbb{R}^3$, its *spherical coordinates* (ρ, θ, ϕ) are given by:



To describe all of $\mathbb{R}^3$ we need:

$$0 \le \rho < \infty,$$

$$0 \le \theta \le 2\pi,$$

$$0 \le \phi \le \pi.$$

Examples

Describe the following surfaces (k is a constant).

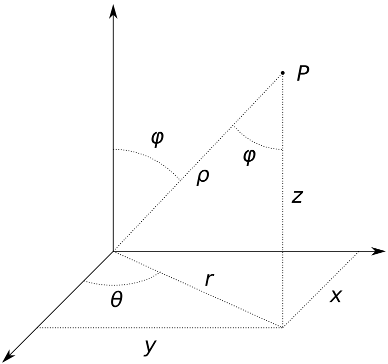
1. <u>$\rho = k$:</u> Sphere of radius k, centered at $(0, 0, 0)$.
2. <u>$\theta = k$:</u> Vertical half-plane "attached to" the z-axis.
3. <u>$\phi = k$:</u> Cone with vertex at $(0, 0, 0)$. Note:
   - The cone opens upward if $0 < \phi < \pi/2$.
   - The cone opens downward if $\pi/2 < \phi < \pi$.
   - The cone is actually the xy-plane if $\phi = \pi/2$.

More Examples

Describe the following regions.

1. $\underline{\rho \leq 2:}$ Solid ball of radius 2, centered at $(0, 0, 0)$.
2. $\underline{\rho \leq 2,\ 0 \leq \phi \leq \pi/2:}$ "Northern hemisphere" of preceding ball.
3. $\underline{2 \leq \rho \leq 3:}$ Spherical shell or "cored" ball, with inner radius 2, outer radius 3, and center $(0, 0, 0)$.
4. $\underline{2 \leq \rho \leq 3,\ 0 \leq \theta \leq \pi/2,\ 0 \leq \phi \leq \pi/2:}$ Portion of the cored ball above in the first octant.
5. $\underline{\rho \leq 3,\ 0 \leq \phi \leq \pi/4:}$ Cone with a spherical "cap" of radius 3.
6. $\underline{\rho \leq 3,\ \pi/4 \leq \phi \leq 3\pi/4:}$ Solid ball with two cones removed.

Relation to Rectangular Coordinates



We see that:

$$\begin{aligned} x &= r\cos\theta \\ &= \rho\sin\phi\cos\theta \\ y &= r\sin\theta \\ &= \rho\sin\phi\sin\theta \\ z &= \rho\cos\phi \\ \rho^2 &= r^2 + z^2 \\ &= x^2 + y^2 + z^2. \end{aligned}$$

To compute integrals in spherical coordinates, we need to relate $dV = dx\,dy\,dz$ to $d\rho\,d\phi\,d\theta$.

Change of Variables in Triple Integrals

If $E, E' \subset \mathbb{R}^3$ are connected by a transformation

$$T(u,v,w) = (x(u,v,w), y(u,v,w), z(u,v,w))$$

(i.e. $T : E' \to E$), one can use a Riemann sum argument to derive the relationship

$$\iiint_E f(x,y,z)\,dV = \iiint_{E'} f(T(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\,dv\,dw,$$

where

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix}$$

is the *Jacobian* of the transformation.

The Jacobian of Spherical Coordinates

For the spherical coordinate transformation

$$\begin{aligned} x &= \rho \sin \phi \cos \theta, \\ y &= \rho \sin \phi \sin \theta, \\ z &= \rho \cos \phi, \end{aligned}$$

we have

$$\frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} = \begin{vmatrix} \sin \phi \cos \theta & -\rho \sin \phi \sin \theta & \rho \cos \phi \cos \theta \\ \sin \phi \sin \theta & \rho \sin \phi \cos \theta & \rho \cos \phi \sin \theta \\ \cos \phi & 0 & -\rho \sin \phi \end{vmatrix} = -\rho^2 \sin \phi,$$

so that

$$\boxed{dV = dx\, dy\, dz = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta}.$$

Examples

Example 3

Evaluate $\iiint_E e^{\sqrt{x^2+y^2+z^2}}\,dV$ where E is the portion of the ball $x^2+y^2+z^2 \le 9$ in the first octant.

Solution. In spherical coordinates we have

$$\begin{aligned}\iiint_E e^{\sqrt{x^2+y^2+z^2}}\,dV &= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^3 e^{\rho}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta\\ &= \int_0^{\pi/2} d\theta \times \int_0^{\pi/2}\sin\phi\,d\phi \times \int_0^3 \rho^2 e^{\rho}\,d\rho\\ &= \frac{\pi}{2}\left(-\cos\phi\Big|_0^{\pi/2}\right)\left((\rho^2-2\rho+2)e^{\rho}\Big|_0^3\right)\end{aligned}$$

$$= \frac{\pi}{2} \cdot 1 \cdot \left(5e^3 - 2\right) = \boxed{\frac{\pi(5e^3 - 2)}{2}}.$$

□

Example 4

Evaluate $\iiint_E y\,dV$, where E is bounded by the xz-plane and the hemispheres $y = \sqrt{9 - x^2 - z^2}$ and $y = \sqrt{16 - x^2 - z^2}$.

Solution. E is the region between the spheres $x^2 + y^2 + z^2 = 9$ and $x^2 + y^2 + z^2 = 16$ with $y \geq 0$, i.e. the "right" half of a cored ball.

This region would be quite difficult to express in rectangular coordinates, but is easy to describe spherically.

In spherical coordinates we have

$$\iiint_E y\,dV = \int_0^\pi \int_0^\pi \int_3^4 \rho \sin\phi \sin\theta \rho^2 \sin\phi \,d\rho\,d\phi\,d\theta$$

$$= \int_0^\pi \sin\theta\,d\theta \times \int_0^\pi \sin^2\phi\,d\phi \times \int_3^4 \rho^3\,d\rho$$

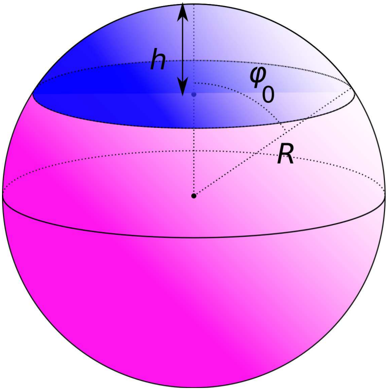
$$= \left(-\cos\theta\Big|_0^\pi\right) \int_0^\pi \frac{1-\cos 2\phi}{2}\,d\phi \left(\frac{\rho^4}{4}\Big|_3^4\right)$$

$$= 2\left(\frac{\phi}{2} - \frac{\sin 2\phi}{4}\Big|_0^\pi\right)\left(\frac{256-81}{4}\right) = \boxed{\frac{175\pi}{4}}.$$

□

Example 5

Find the volume of the spherical "cap" of height h of a sphere of radius R.



Solution. Position the sphere at the origin with the cap at the north pole.

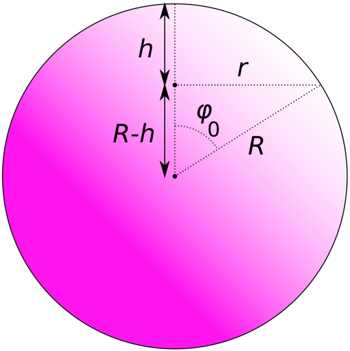
Let ϕ_0 denote the angle that the edge of the cap makes with the north pole.

The volume of the cap is then the volume of the spherical region E described by $0 \leq \rho \leq R$ and $0 \leq \phi \leq \phi_0$, *minus* the volume of its lower conical portion (which can be computed from elementary geometry).

We have

$$\begin{aligned}
\mathrm{Vol}(E) &= \iiint_E dV = \int_0^{2\pi} \int_0^{\phi_0} \int_0^R \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \\
&= \int_0^{2\pi} d\theta \times \int_0^{\phi_0} \sin\phi \, d\phi \times \int_0^R \rho^2 \, d\rho \\
&= 2\pi \left(-\cos\phi \Big|_0^{\phi_0} \right) \left(\frac{\rho^3}{3} \Big|_0^R \right) = \frac{2\pi R^3}{3} (1 - \cos\phi_0) .
\end{aligned}$$

Looking at a cross section through the north pole we find that



$$\cos\phi_0 = \frac{R-h}{R} = 1 - \frac{h}{R}.$$

Thus

$$\mathrm{Vol}(E) = \frac{2\pi R^3}{3}(1-\cos\phi_0) = \frac{2\pi R^2 h}{3}.$$

The same diagram shows that the radius of the cone satisfies

$$r^2 + (R-h)^2 = R^2 \Rightarrow r^2 = R^2 - (R-h)^2 = 2Rh - h^2,$$

so that its volume is given by

$$\frac{\pi r^2(R-h)}{3} = \frac{\pi(2Rh-h^2)(R-h)}{3}.$$

It now follows that the volume of the cap is

$$\frac{2\pi R^2 h}{3} - \frac{\pi(2Rh-h^2)(R-h)}{3} = \boxed{\frac{\pi h^2}{3}(3R-h)}.$$

□