Vector Valued Functions: Parametric Curves

Ryan C. Daileda

Trinity University

Calculus III

Introduction

In order to continue our study of integration, we need to be able to describe curves in $\mathbb{R}^2$ and $\mathbb{R}^3$.

One convenient way to describe the points on a given curve is to specify x, y (and z) as functions of some parameter t.

It will also be convenient to encapsulate this parametric representation in a vector $\mathbf{r}(t)$.

This gives rise to single-variable vector valued functions, which we will now begin to study.

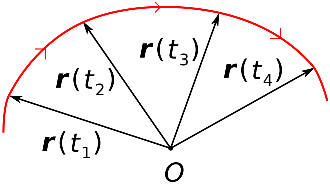
Vector Functions of One Variable

A *single-variable vector function* or *parametric curve* has the form

$$\mathbf{r}(t) = \langle x(t), y(t) \rangle \quad \text{or} \quad \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle,$$

where $x(t)$, $y(t)$ and $z(t)$ are ordinary real-valued functions of one (real) variable.

We treat $\mathbf{r}(t)$ as a position vector whose tip traces out an oriented curve as t varies.



$$t_1 < t_2 < t_3 < t_4$$

Examples

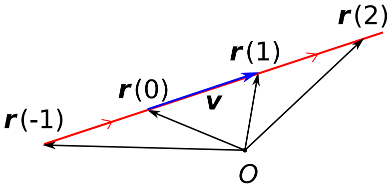
Example 1

Describe the graph of the vector function $\mathbf{r}(t) = \langle -t, 1+t, 2+3t \rangle$.

Solution. If we write

$$\mathbf{r}(t) = \langle 0, 1, 2 \rangle + t\langle -1, 1, 3 \rangle,$$

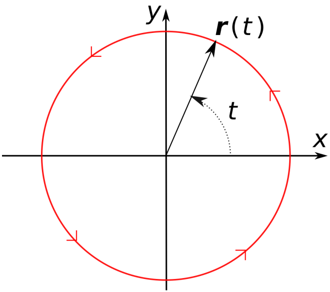
we recognize the vector expression for the line through $(0, 1, 2)$ with direction $\mathbf{v} = \langle -1, 1, 3 \rangle$.



Example 2

Describe the graph of $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$.

Solution. We recognize $\mathbf{r}(t)$ as the position vector of the point with polar coordinates $r = 1$ and $\theta = t$.



Therefore the tip of $\mathbf{r}(t)$ describes the unit circle oriented counterclockwise.

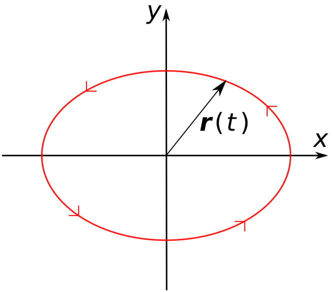
Example 3

Describe the graph of $\mathbf{r}(t) = \langle a\cos t, b\sin t\rangle$, where $a, b > 0$.

Solution. We simply notice that $x = a\cos t$, $y = b\sin t$ satisfy the equation

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \quad \Leftrightarrow \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

which is the equation of an ellipse with axes a and b.



The parameter t is *not* the polar coordinate angle θ, but the two are related by the equation

$$\theta = \arctan\left(\frac{b}{a}\tan t\right).$$

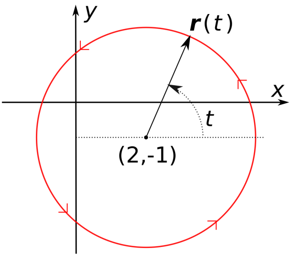
Example 4

Describe the graph of $\mathbf{r}(t) = \langle 2 + 3\cos t, -1 + 3\sin t \rangle$.

Solution. We note that

$$\mathbf{r}(t) = \underbrace{\langle 2, -1 \rangle}_{\text{translation}} + \underbrace{\langle 3\cos t, 3\sin t \rangle}_{\text{circle}},$$

so that the graph is a counterclockwise circle of radius 3 centered at $(2, -1)$.



The parameter t still represents the angle relative to the horizontal.

Example 5

Describe the graph of $\mathbf{r}(t) = \langle x_0 + a\cos t, y_0 + b\sin t \rangle$, where $a, b > 0$.

Solution. As in the previous example, we have

$$\mathbf{r}(t) = \underbrace{\langle x_0, y_0 \rangle}_{\text{translation}} + \underbrace{\langle a\cos t, b\sin t \rangle}_{\text{ellipse}},$$

so that the graph is a counterclockwise ellipse with axes a and b, "centered" at (x_0, y_0).

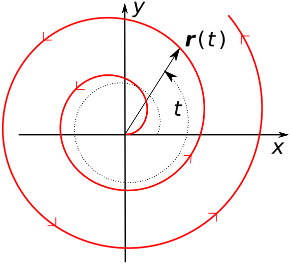
Example 6

Describe the graph of $\mathbf{r}(t) = \langle t \cos t, t \sin t \rangle$.

Solution. In this case $\mathbf{r}(t)$ is the position vector of the point with polar coordinates $r = t$ and $\theta = t$.

So as t increases, $\mathbf{r}(t)$ rotates around the origin while its magnitude steadily grows.

This means that its graph will be a counterclockwise spiral.



Example 7

Describe the curve $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$.

Solution. Here $\mathbf{r}(t)$ is the position vector of a point in $\mathbb{R}^3$ with cylindrical coordinates $r = 1$, $\theta = t$ and $z = t$.

$\mathbf{r}(t)$ is therefore confined to the cylinder of radius 1 along the z-axis.

As t increases, $\theta = t$ rotates around the z-axis while $z = t$ steadily increases.

The graph is therefore a counterclockwise helix along the z-axis. See Maple diagram.

Example 8

Describe the graph of $\mathbf{r}(t) = \langle \cos t \sin 4t, \sin t \sin 4t, \cos 4t \rangle$.

Solution. We recognize $\mathbf{r}(t)$ as the position vector of a point in $\mathbb{R}^3$ with spherical coordinates $\rho = 1$, $\theta = t$ and $\phi = 4t$.

The graph of $\mathbf{r}(t)$ is therefore confined to the unit sphere $x^2 + y^2 + z^2 = 1$.

As t increases, $\theta = t$ wraps steadily around the sphere, while $\phi = 4t$ moves between the north and south poles four times as fast.

See Maple diagram.

Example 9

Describe the graph of $\mathbf{r}(t) = \langle \cos 10t \sin t, \sin 10t \sin t, \cos t \rangle$.

Solution. This time, the spherical coordinates of $\mathbf{r}(t)$ are $\rho = 1$, $\theta = 10t$ and $\phi = t$.

The graph again is confined to the unit sphere $x^2 + y^2 + z^2 = 1$.

But this time as t increases, $\phi = t$ moves steadily between the north and south poles, while $\theta = 10t$ wraps around the sphere ten times as fast.

The result is a "spherical tornado." See Maple diagram.

Example 10

Parametrize the line segment from $(1,3)$ to $(4,-2)$.

Solution. The direction of the line segment in question is

$$\mathbf{v} = \langle 4,-2\rangle - \langle 1,3\rangle = \langle 3,-5\rangle.$$

So the entire line through $(1,3)$ with direction $\mathbf{v}$ is

$$\mathbf{r}(t) = \langle 1,3\rangle + t\langle 3,-5\rangle = \boxed{\langle 1+3t, 3-5t\rangle}.$$

Since $\mathbf{r}(0) = \langle 1,3\rangle$ while $\mathbf{r}(1) = \langle 4,-2\rangle$, we can describe the segment alone by requiring that

$$\boxed{0 \le t \le 1}.$$

Example 11

Parametrize the portion of the circle $x^2 + y^2 = 9$ that lies above the x-axis, oriented counterclockwise.

Solution. In polar coordinates this portion of the circle is described by $r = 3$ and $0 \leq \theta \leq \pi/2$.

We can therefore parametrize by taking $t = \theta$:

$$\boxed{\mathbf{r}(t) = \langle 3\cos t, 3\sin t \rangle},$$

for $\boxed{0 \leq t \leq \pi/2}$.

Because θ increases in the counterclockwise direction, this will have the desired orintation.