Derivatives of Vector Valued Functions: Tangent Vectors, Arc Length and Curvature

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Calculus III

Introduction

For the purpose of line integration it will be important to be able to compute and understand the derivatives of vector functions.

As with partial differentiation and multiple integration, the mechanics of differentiation in this setting will be familiar from Calculus I.

However, we will see several new concepts: tangent vectors, arc length and curvature.

Although we won't make heavy use of curvature later, it is an interesting concept worth studying here.

Derivatives of Vector Valued Functions

Definition

The *derivative* of $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ is

$$\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle.$$

Remark. For vector functions in 2D, we simply omit the third component.

Examples.

1. The derivative of $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$ is $\mathbf{r}'(t) = \langle -\sin t, \cos t \rangle$.
2. The derivative of $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ is $\mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle$.

Geometry of the Derivative

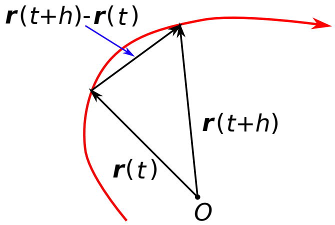
Recall the limit definition of the derivative:

$$f'(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}.$$

Applying this in each component, we find that

$$\begin{aligned} \mathbf{r}'(t) &= \lim_{h \to 0} \frac{1}{h} \langle x(t+h) - x(t), y(t+h) - y(t), z(t+h) - z(t) \rangle \\ &= \lim_{h \to 0} \frac{1}{h} (\mathbf{r}(t+h) - \mathbf{r}(t)). \end{aligned}$$

Graphically we have:



The quantity $\mathbf{r}(t+h) - \mathbf{r}(t)$ therefore represents a (small) displacement along the graph of $\mathbf{r}(t)$.

Hence the difference quotient $\frac{1}{h}(\mathbf{r}(t+h) - \mathbf{r}(t))$ represents the average velocity of $\mathbf{r}(t)$.

Taking the limit as $h \to 0$, $\mathbf{r}(t+h) - \mathbf{r}(t)$ becomes tangent to $\mathbf{r}(t)$, so $\frac{1}{h}(\mathbf{r}(t+h) - \mathbf{r}(t))$ does as well.

We conclude that

$$\boxed{\mathbf{r}'(t) \text{ is tangent to the graph of } \mathbf{r}(t),}$$

$\mathbf{r}'(t)$ gives the orientation of $\mathbf{r}(t)$, and that

$\mathbf{r}'(t)$ represents the instantaneous velocity
of a particle moving along $\mathbf{r}(t)$.

Remark. This means that we do *not* interpret $\mathbf{r}'(t)$ as another parametric curve, but as a vector whose tail is at the tip of the position vector $\mathbf{r}(t)$.

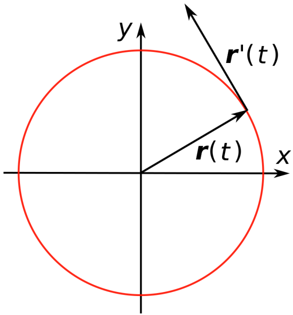
Examples

Example 1

Sketch the curve $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$ and its derivative.

Solution. We know that $\mathbf{r}(t)$ represents the unit circle, oriented counterclockwise.

Since $\mathbf{r}'(t) = \langle -\sin t, \cos t \rangle$ and $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$, we have the following diagram:



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Example 2

Find the tangent line to $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ at the point $(1, 1, 1)$.

Solution. We are at the point $(1, 1, 1)$ when $t = 1$.

Therefore $\mathbf{r}'(1)$ gives the direction of the tangent line:

$$\mathbf{r}'(1) = \langle 1, 2t, 3t^2 \rangle \Big|_{t=1} = \langle 1, 2, 3 \rangle.$$

Hence the tangent line is given by

$$\boxed{\ell(t) = \langle 1, 1, 1 \rangle + t\langle 1, 2, 3 \rangle = \langle 1 + t, 1 + 2t, 1 + 3t \rangle}.$$

$\square$

Example 3

Show that

$$\frac{d}{dt}|\mathbf{r}(t)| = \frac{1}{|\mathbf{r}(t)|}\mathbf{r}(t)\cdot\mathbf{r}'(t)$$

and interpret geometrically.

Solution. We have $|\mathbf{r}(t)|^2 = \mathbf{r}(t)\cdot\mathbf{r}(t)$.

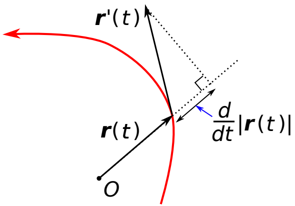
One can show that the usual laws of differentiation hold for vector functions, so that we have

$$\begin{aligned}\frac{d}{dt}|\mathbf{r}(t)|^2 &= \frac{d}{dt}(\mathbf{r}(t)\cdot\mathbf{r}(t))\\ 2|\mathbf{r}(t)|\frac{d}{dt}|\mathbf{r}(t)| &= \mathbf{r}(t)\cdot\mathbf{r}'(t)+\mathbf{r}'(t)\cdot\mathbf{r}(t)\\ &= 2\mathbf{r}(t)\cdot\mathbf{r}'(t).\end{aligned}$$

If we divide both sides by $2|\mathbf{r}(t)|$,we get

$$\frac{d}{dt}\,|\mathbf{r}(t)| = \frac{1}{|\mathbf{r}(t)|}\mathbf{r}(t)\cdot\mathbf{r}'(t) = \operatorname{comp}_{\mathbf{r}(t)}\mathbf{r}'(t).$$

That is, the rate of change of $|\mathbf{r}(t)|$ is the scalar projection of $\mathbf{r}'(t)$ onto $\mathbf{r}(t)$.
This says that as we move along the graph of $\mathbf{r}(t)$, the rate at which we move away from the origin is $\operatorname{comp}_{\mathbf{r}(t)}\mathbf{r}'(t)$.



Arc Length

Suppose we are given a curve $\mathbf{r}(t)$ defined for $t \in [a, b]$.

If we subdivide $[a, b]$ as

$$t_0 = a < t_1 < t_2 < \ldots < t_{n-1} < t_n = b,$$

then the approximate length of the graph of $\mathbf{r}(t)$ is

$$\sum_{i=1}^{n} |\mathbf{r}(t_i) - \mathbf{r}(t_{i-1})| = \sum_{i=1} \left| \frac{1}{\Delta t_i} (\mathbf{r}(t_i) - \mathbf{r}(t_{i-1})) \right| \Delta t_i \approx \sum_{i=1}^{n} |\mathbf{r}'(t_{i-1})| \Delta t_i,$$

which is a Riemann sum.

As $\Delta t \to 0$, both approximations improve indefinitely, so that

$$\boxed{\text{Arc length of } \mathbf{r}(t) = \int_a^b |\mathbf{r}'(t)|\, dt.}$$

Example 4

Find the length of the spiral $\mathbf{r}(t) = \langle t\cos t, t\sin t\rangle$ for $0 \le t \le 2\pi$.

Solution. We have

$$\mathbf{r}'(t) = \langle \cos t - t\sin t, \sin t + t\cos t\rangle,$$

so that

$$|\mathbf{r}'(t)| = \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2}$$

$$= \sqrt{\cos^2 t - 2t \sin t \cos t + t^2 \sin^2 t + \sin^2 t + 2t \sin t \cos t + t^2 \cos^2 t}$$
$$= \sqrt{1 + t^2}.$$

Hence the arc length is given by

$$\int_0^{2\pi} \sqrt{1 + t^2}\, dt.$$

To evaluate this integral we make the trig. substitution $t = \tan\theta$, $dt = \sec^2\theta\, d\theta$:

$$\int_0^{2\pi} \sqrt{1 + t^2}\, dt = \int_0^{\arctan 2\pi} \sqrt{1 + \tan^2\theta}\, \sec^2\theta\, d\theta$$
$$= \int_0^{\arctan 2\pi} \sec^3\theta\, d\theta.$$

This last integral is notoriously difficult to evaluate. Integrating by parts with $u = \sec\theta$ and $dv = \sec^2\theta\,d\theta$, we find that

$$\int_0^{\arctan 2\pi} \sec^3\theta\,d\theta = \sec\theta\tan\theta\Big|_0^{\arctan 2\pi} - \int_0^{\arctan 2\pi} \tan^2\theta\sec\theta\,d\theta$$

$$= 2\pi\sqrt{1+4\pi^2} - \int_0^{\arctan 2\pi} (\sec^2\theta - 1)\sec\theta\,d\theta$$

$$= 2\pi\sqrt{1+4\pi^2} - \int_0^{\arctan 2\pi} \sec^3\theta\,d\theta + \int_0^{\arctan 2\pi} \sec\theta\,d\theta$$

$$= 2\pi\sqrt{1+4\pi^2} + \ln|\sec\theta + \tan\theta|\Big|_0^{\arctan 2\pi} - \int_0^{\arctan 2\pi} \sec^3\theta\,d\theta$$

$$= 2\pi\sqrt{1+4\pi^2} + \ln\left(2\pi + \sqrt{1+4\pi^2}\right) - \int_0^{\arctan 2\pi} \sec^3\theta\,d\theta.$$

Solving for the integral we finally obtain

$$\begin{aligned}\text{Arc length} &= \int_0^{\arctan 2\pi} \sec^3\theta \, d\theta \\ &= \boxed{\pi\sqrt{1+4\pi^2} + \frac{1}{2}\ln\left(2\pi + \sqrt{1+4\pi^2}\right)}.\end{aligned}$$

$\square$

The Arc Length Function

Given a parametric curve $\mathbf{r}(t)$, $a \leq t \leq b$, we define its *arc length function* to be

$$s(t) = \int_a^t |\mathbf{r}'(\tau)| \, d\tau.$$

$s(t)$ gives the length of the graph from $\mathbf{r}(a)$ to $\mathbf{r}(t)$.

Example 5

Compute the arc length function for $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$, starting at $t = 0$.

Solution. We have $\mathbf{r}'(t) = \langle -\sin t, \cos t, 1 \rangle$, so that

$$|\mathbf{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}.$$

Curvature

Hence the arc length function is

$$s(t) = \int_0^t \sqrt{2}\, d\tau = \boxed{\sqrt{2}t}.$$

$\square$

Definition

We define the *curvature* of $\mathbf{r}(t)$ to be

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right|,$$

where $\mathbf{T} = \mathbf{r}'(t)/|\mathbf{r}'(t)|$ is the unit tangent vector and s is the arc length function.

Remark. In words, the curvature is the rate at which the tangent direction changes with respect to arc length.

We have seen that arc length functions can be difficult to compute, so it would be nice to have a way to compute κ directly. If we differentiate $s(t)$ and apply FTOC we obtain

$$\frac{ds}{dt} = |\mathbf{r}'(t)|,$$

so that $\mathbf{r}'(t) = |\mathbf{r}'(t)|\mathbf{T} = \frac{ds}{dt}\mathbf{T}$.
Now use the product rule to obtain

$$\mathbf{r}''(t) = \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\mathbf{T}'.$$

Since $\mathbf{T} \times \mathbf{T} = \mathbf{0}$, taking the cross product with $\mathbf{r}'(t) = \frac{ds}{dt}\mathbf{T}$ yields

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \left(\frac{ds}{dt}\right)^2 \mathbf{T} \times \mathbf{T}'$$

Since $|\mathbf{T}| = 1$, our earlier example implies that

$$0 = \frac{d}{dt}|\mathbf{T}| = \frac{\mathbf{T} \cdot \mathbf{T}'}{|\mathbf{T}|} \;\Rightarrow\; \mathbf{T} \cdot \mathbf{T}' = 0.$$

This means that $\mathbf{T}$ and $\mathbf{T}'$ are orthogonal, so that

$$|\mathbf{T} \times \mathbf{T}'| = |\mathbf{T}| \cdot |\mathbf{T}'| \sin\theta = |\mathbf{T}'|.$$

Thus

$$|\mathbf{r}' \times \mathbf{r}''| = \left(\frac{ds}{dt}\right)^2 |\mathbf{T} \times \mathbf{T}'| = \left(\frac{ds}{dt}\right)^2 |\mathbf{T}'| = |\mathbf{r}'|^2 |\mathbf{T}'|$$

But by the chain rule

$$|\mathbf{T}'| = \left|\frac{d\mathbf{T}}{dt}\right| = \left|\frac{d\mathbf{T}}{ds}\frac{ds}{dt}\right| = \kappa |\mathbf{r}'|.$$

Putting these last two equations together we obtain

$$|\mathbf{r}' \times \mathbf{r}''| = \kappa |\mathbf{r}'|^3 \quad \Rightarrow \quad \boxed{\kappa = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}}.$$

Example 6

Find the curvature of the circle $\mathbf{r}(t) = \langle R\cos t, R\sin t\rangle$.

Solution. We have $\mathbf{r}' = \langle -R\sin t, R\cos t\rangle$ and $\mathbf{r}'' = \langle -R\cos t, -R\sin t\rangle$, so that

$$\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} -R\sin t & R\cos t \\ -R\cos t & -R\sin t \end{vmatrix} \mathbf{k} = R^2\mathbf{k}.$$

Since $|\mathbf{r}'| = R$, we find that

$$\kappa = \frac{R^2}{R^3} = \boxed{\frac{1}{R}},$$

i.e. the curvature of a circle of radius R is constant and inversely proportional to its radius. $\square$

Example 7

Find the curvature of $\mathbf{r}(t) = \langle t, 4t^{3/2}, -t^2 \rangle$.

Solution. We have $\mathbf{r}' = \langle 1, 6t^{1/2}, -2t \rangle$ and $\mathbf{r}'' = \langle 0, 3/t^{1/2}, -2 \rangle$, so that

$$\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 6t^{1/2} & -2t \\ 0 & 3/t^{1/2} & -2 \end{vmatrix} = -6t^{1/2}\mathbf{i} + 2\mathbf{j} + \frac{3}{t^{1/2}}\mathbf{k}.$$

Therefore

$$|\mathbf{r}' \times \mathbf{r}''| = \sqrt{36t + 4 + 9/t}$$

and

$$|\mathbf{r}'| = \sqrt{1 + 36t + 4t^2},$$

so that

$$\kappa = \frac{\sqrt{36t + 4 + 9/t}}{(1 + 36t + 4t^2)^{3/2}} = \boxed{\frac{1}{1 + 36t + 4t^2}\sqrt{\frac{36t^2 + 4t + 9}{4t^3 + 36t^2 + t}}}.$$

$\square$