Line Integrals

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Calculus III

Introduction

Today we will develop the theory of *line integrals* of functions.

This will allow us to integrate functions of several variables along curves (rather than regions of the plane or space).

We will develop a number of line integrals, depending on how we choose to measure the distance between points.

Our eventual goal is to study the integrals of *1-forms* or *vector fields*, which can be expressed and understood in terms of the integrals developed here.

Line Integrals

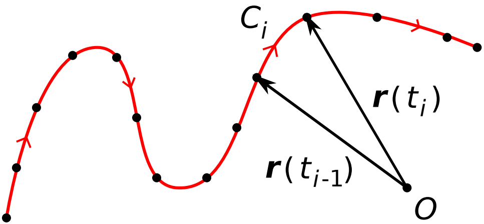
Suppose we are given an oriented curve C parametrized by $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ with $t \in [a, b]$, and a function $f(x, y)$ whose domain includes C.

Goal. Use the "usual" procedure to integrate f along C.

We first subdivide $[a, b]$:

$$a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b.$$

This divides C into subarcs C_i parametrized by $\mathbf{r}(t)$ with $t \in [t_{i-1}, t_i]$.

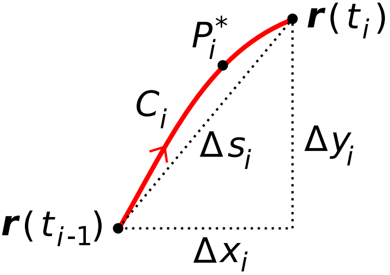


We choose sample points $P_i^* \in C_i$ and construct the Riemann sum

$$\sum_{i=1}^{n} f(P_i^*)m(C_i),$$

where $m(C_i)$ is some geometric "measurement" of C_i.

We will consider three different choices for $m(C_i)$:



- $m(C_i) = \Delta x_i = x(t_i) - x(t_{i-1})$
- $m(C_i) = \Delta y_i = y(t_i) - y(t_{i-1})$
- $m(C_i) = \Delta s_i = |\mathbf{r}(t_i) - \mathbf{r}(t_{i-1})|$

Each choice of m leads to a different type of line integral.

We define:

$$\int_C f(x,y)\,dx = \lim_{\Delta t \to 0} \sum_{i=1}^{n} f(P_i^*)\Delta x_i,$$

$$\int_C f(x,y)\,dy = \lim_{\Delta t \to 0} \sum_{i=1}^{n} f(P_i^*)\Delta y_i,$$

$$\int_C f(x,y)\,ds = \lim_{\Delta t \to 0} \sum_{i=1}^{n} f(P_i^*)\Delta s_i,$$

the *line integrals of* f *along* C *with respect to* x, y *and arc length*.

Interpreting and Evaluating Line Integrals

The line integral $\int_C f(x,y)\,ds$ represents the signed area of the "fence" between C and the graph of f. See Maple diagram.

The line integrals $\int_C f(x,y)\,dx$ and $\int_C f(x,y)\,dy$ are more easily understood in the context of vector fields.

To compute line integrals we observe that

$$\Delta x_i = x(t_i) - x(t_{i-1}) \approx x'(t_i)\Delta t_i,$$

$$\Delta y_i = y(t_i) - y(t_{i-1}) \approx y'(t_i)\Delta t_i,$$

$$\Delta \mathbf{r}_i = |\mathbf{r}(t_i) - \mathbf{r}(t_{i-1})| \approx |\mathbf{r}'(t_i)|\Delta t_i,$$

and that these approximations become more and more accurate as $\Delta t \to 0$.

Since $P_i^* = (x(t_i^*), y(t_i^*))$ for some $t_i^* \in [t_{i-1}, t_i]$,

$$\int_C f(x,y)\,dx = \lim_{\Delta t \to 0} \sum_{i=1}^{n} f(P_i^*)\Delta x_i$$

$$= \lim_{\Delta t \to 0} \sum_{i=1}^{n} f(x(t_i^*), y(t_i^*)) x'(t_i)\Delta t_i = \int_a^b f(x(t), y(t))\, x'(t)\, dt.$$

Similarly we have

$$\int_C f(x,y)\,dy = \int_a^b f(x(t), y(t))\, y'(t)\, dy,$$

$$\int_C f(x,y)\,ds = \int_a^b f(\mathbf{r}(t))\, |\mathbf{r}'(t)|\, dt.$$

Remarks

- These formulae can easily be remembered through the following "substitution" rules:

$$x = x(t), \;\; y = y(t),$$
$$dx = x'(t)\,dt, \;\; dy = y'(t)\,dt,$$
$$ds = \sqrt{dx^2 + dy^2} = \sqrt{x'(t)^2 + y'(t)^2}\,dt$$

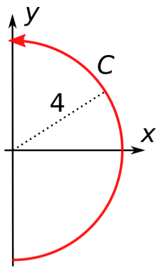
- Strictly speaking, we require $\mathbf{r}'(t) \neq 0$ throughout $[a, b]$ so that the parametrization doesn't "double back" on C.
- With this restriction on $\mathbf{r}'(t)$, one can show that $\int_C$ is *independent* of the parametrization of C chosen.

Example

Example 1

Evaluate $\int_C xy^2\,dx$, $\int_C xy^2\,dy$ and $\int_C xy^2\,ds$,where C is the semicircle of radius 4 centered at the origin, from $(0,-4)$ to $(0,4)$.

Solution.



We parametrize C using polar coordinates:

$$\mathbf{r}(t) = \langle \underbrace{4\cos t}_{x(t)}, \underbrace{4\sin t}_{y(t)} \rangle, \quad -\pi/2 \le t \le \pi/2.$$

We have

$$dx = x'(t)\,dt = \;-4\sin t\,dt,$$
$$dy = y'(t)\,dt = 4\cos t\,dt,$$
$$ds = \sqrt{dx^2 + dy^2} = \sqrt{16}\,dt = 4\,dt.$$

Therefore:

$$\int_C xy^2\,dx = \int_{-\pi/2}^{\pi/2} (4\cos t)(4\sin t)^2(-4\sin t)\,dt$$
$$= -256\int_{-\pi/2}^{\pi/2} \underbrace{\sin^3 t\cos t}_{\text{odd}}\,dt = \boxed{0},$$
$$\int_C xy^2\,dy = \int_{-\pi/2}^{\pi/2} (4\cos t)(4\sin t)^2(4\cos t)\,dt$$
$$= 256\int_{-\pi/2}^{\pi/2} \underbrace{\sin^2 t\cos^2 t}_{\text{even}}\,dt$$

$$= 512\int_0^{\pi/2}\frac{1-\cos 2t}{2}\frac{1+\cos 2t}{2}\,dt$$

$$= 128\int_0^{\pi/2} 1-\cos^2 2t\,dt = 128\int_0^{\pi/2} 1-\frac{1+\cos 4t}{2}\,dt$$

$$= 64\int_0^{\pi/2} 1-\cos 4t\,dt = 64\left(t-\frac{\sin 4t}{4}\Big|_0^{\pi/2}\right) = \boxed{32\pi},$$

$$\int_C xy^2\,ds = \int_{-\pi/2}^{\pi/2}\underbrace{(4\cos t)(4\sin t)^2}_{\text{even}}4\,dt$$

$$= 512\int_0^{\pi/2}\sin^2 t\cos t\,dt = 512\frac{\sin^3 t}{3}\Big|_0^{\pi/2} = \boxed{\frac{512}{3}}.$$

□

Remarks

- We have:

$$\int_C ds = \int_a^b |\mathbf{r}'(t)|\,dt = \text{Arc length of } C,$$

$$\int_C dx = \int_a^b x'(t)\,dt = x(t)\Big|_a^b = x(\text{end}) - x(\text{beg.}),$$

$$\int_C dy = \int_a^b y'(t)\,dt = y(t)\Big|_a^b = y(\text{end}) - y(\text{beg.}).$$

- We define

$$\int_C P(x,y)\,dx + Q(x,y)\,dy = \int_C P(x,y)\,dx + \int_C Q(x,y)\,dy.$$

Remarks (Cont.)

- Given an oriented curve C, we let $-C$ denote the same path, with the opposite orientation. We have:

$$\int_{-C} f(x,y)\,dx = -\int_C f(x,y)\,dx,$$
$$\int_{-C} f(x,y)\,dy = -\int_C f(x,y)\,dy,$$
$$\int_{-C} f(x,y)\,ds = \int_C f(x,y)\,ds.$$

- If C is made up of successive pieces $C_1, C_2, \ldots$, we write $C = C_1 + C_2 + \cdots$, and we have

$$\int_{C_1+C_2+\cdots} = \int_{C_1} + \int_{C_2} + \cdots$$

Examples

Example 2

Evaluate $\int_C xy\,dx + (x-y)\,dy$, where C consists of the line segments connecting $(0,0)$ to $(2,0)$ to $(3,2)$.

Solution. We integrate on each segment separately, then add the results.

For convenience we set $\omega = xy\,dx + (x-y)\,dy$.

On the first segment C_1 we have $y = 0$ and $dy = 0\,dt$. Thus

$$\int_{C_1} \omega = \int_{C_1} \underbrace{xy}_{0}\,dx + (x-y)\,\underbrace{dy}_{0} = 0.$$

We parametrize the second segment C_2 in the usual way:

$$\mathbf{r}(t) = \langle 2, 0\rangle + t\langle 3-2, 2-0\rangle = \langle 2+t, 2t\rangle, \ \ 0 \le t \le 1.$$

Thus $x = 2+t$, $y = 2t$, $dx = dt$ and $dy = 2\,dt$, so that

$$\int_{C_2} \omega = \int_{C_2} xy\,dx + (x-y)\,dy = \int_0^1 (2+t)(2t) + (2-t)2\,dt$$
$$= \int_0^1 2t^2 + 2t + 4\,dt = \frac{2}{3} + 1 + 4 = \frac{17}{3}.$$

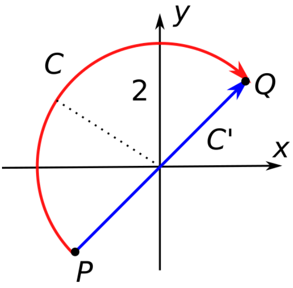
Therefore

$$\int_C \omega = \int_{C_1+C_2} \omega = \int_{C_1} \omega + \int_{C_2} \omega = 0 + \frac{17}{3} = \boxed{\frac{17}{3}}.$$

□

Example 3

Let $P = (-\sqrt{2}, -\sqrt{2})$ and $Q = (\sqrt{2}, \sqrt{2})$. Compare $\int_C x\,dy$ and $\int_{C'} x\,dy$, where C and C' are the paths from P to Q shown below.



Solution. We parametrize $-C$ using polar coordinates:

$$\mathbf{r}(t) = \langle 2\cos t, 2\sin t\rangle, \quad \pi/4 \le t \le 5\pi/4.$$

This yields $x = 2\cos t$ and $dy = 2\cos t\,dt$, so that

$$\int_C x\,dy = -\int_{-C} x\,dy = -\int_{\pi/4}^{5\pi/4} (2\cos t)(2\cos t)\,dt$$
$$= -4\int_{\pi/4}^{5\pi/4} \frac{1+\cos 2t}{2}\,dt = -2\left(t + \frac{\sin 2t}{2}\Big|_{\pi/4}^{5\pi/4}\right)$$
$$= \boxed{-2\pi}.$$

On the other hand, C' is given by

$$\mathbf{r}(t) = \langle -\sqrt{2}, -\sqrt{2}\rangle + t\langle \sqrt{2} - (-\sqrt{2})\sqrt{2} - (-\sqrt{2})\rangle$$
$$= \sqrt{2}\langle -1+2t, -1+2t\rangle,\ \ 0 \le t \le 1,$$

so that $x = \sqrt{2}(-1+2t)$ and $dy = 2\sqrt{2}t$.

Thus

$$\begin{aligned}\int_{C'} x\,dy &= \int_0^1 \sqrt{2}(-1+2t)2\sqrt{2}\,dt \\ &= 4\int_0^1 -1+2t\,dt = 4\left(-t+t^2\Big|_0^1\right) \\ &= 4(-1+1) = \boxed{0}.\end{aligned}$$

□

Moral. When integrating between two points, $\int_C$ *depends on the choice of path* (in general).

Line Integrals in 3D

Given a function $f(x, y, z)$ and an oriented curve C parametrized by $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ with $a \leq t \leq b$, the analogous Riemann sum procedure yields

$$\int_C f(x, y, z)\, dx = \int_a^b f(x(t), y(t), z(t))\, x'(t)\, dt,$$

$$\int_C f(x, y, z)\, dy = \int_a^b f(x(t), y(t), z(t))\, y'(t)\, dt,$$

$$\int_C f(x, y, z)\, dz = \int_a^b f(x(t), y(t), z(t))\, z'(t)\, dt,$$

$$\int_C f(x, y, z)\, ds = \int_a^b f(\mathbf{r}(t))\, |\mathbf{r}'(t)|\, dt.$$