Vector Fields

Ryan C. Daileda

Trinity University

Calculus III

Introduction

Today we will introduce another kind of vector-valued function, namely vector fields.

Vector fields and their line integrals are important in applications.

A vector field is simply a “repackaged” differential 1-form, and this is where we will begin.

Differential 1-Forms

Definition

An expression of the form

$$\omega = P(x,y)\,dx + Q(x,y)\,dy = P\,dx + Q\,dy$$

is called a *(differential) 1-form*.

The *domain* of a 1-form ω is the intersection of the domains of P and Q.

We say that ω is continuous, differentiable, etc. provided P and Q are both continuous, differentiable, etc.

From now on we assume ω is always C^1, i.e. the first order partial derivatives of P and Q exist and are continuous.

Integration of 1-Forms

Differential 1-forms associate numbers to oriented curves.

Specifically, every 1-form ω with domain D yields a linear function

$$E_\omega : \{\text{oriented curves in } D\} \to \mathbb{R},$$

defined by $E_\omega(C) = \int_C \omega$.

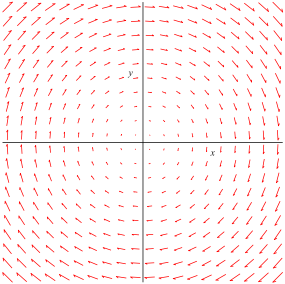
Suppose $\omega = P\,dx + Q\,dy$ and C is given by $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ with $a \le t \le b$. Then by definition

$$\int_C \omega = \int_a^b P(x(t), y(t))\, x'(t) + Q(x(t), y(t))\, y'(t)\, dt$$

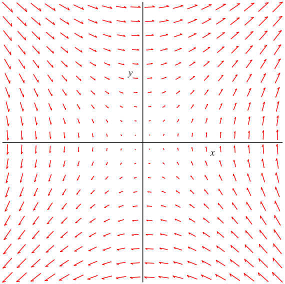
$$= \int_a^b \underbrace{\langle P, R\rangle}_{\mathbf{F}} \cdot \underbrace{\mathbf{r}'(t)\, dt}_{d\mathbf{r}} = \int_C \mathbf{F} \cdot d\mathbf{r}.$$

The vector-valued function $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y)\rangle = P\mathbf{i} + Q\mathbf{j}$ is called a *vector field*.

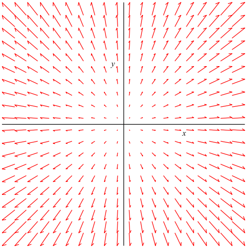
We visualize a vector field $\mathbf{F}$ by drawing the vector $\mathbf{F}(x, y)$ with its tail at (x, y).



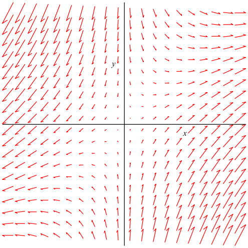
$\mathbf{F} = \langle y, -x\rangle$



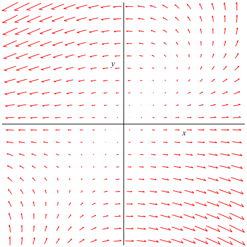
$\mathbf{F} = \langle y, x\rangle$



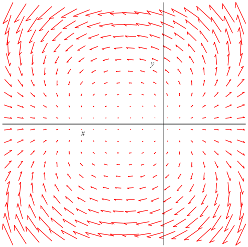
$\mathbf{F} = \langle x, y \rangle$



$\mathbf{F} = \langle x, x - y \rangle$



$\mathbf{F} = \langle x - y, xy \rangle$



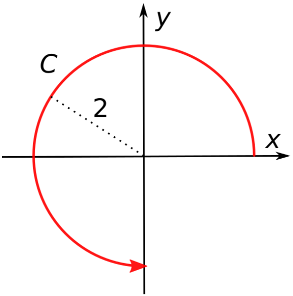
$\mathbf{F} = \langle x^2 + x - y^2, 2xy + y \rangle$

Example

Example 1

If $\mathbf{F} = \langle x - y, xy \rangle$, compute $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is $x^2 + y^2 = 4$ from $(2, 0)$ to $(0, -2)$, counterclockwise.

Solution.



We parametrize C using polar coordinates:

$$\mathbf{r}(t) = \langle 2\cos t, 2\sin t \rangle,$$

with $0 \leq t \leq 3\pi/2$.

We have $x = 2\cos t, y = 2\sin t$ and $\mathbf{r}'(t) = \langle -2\sin t, 2\cos t\rangle$, so that

$$\begin{aligned}
&\int_C \mathbf{F}\cdot d\mathbf{r} \\
&= \int_0^{3\pi/2} \langle 2\cos t - 2\sin t, (2\cos t)(2\sin t)\rangle \cdot \langle -2\sin t, 2\cos t\rangle\, dt \\
&= \int_0^{3\pi/2} -4\sin t\cos t + 4\sin^2 t + 8\cos^2 t \sin t\, dt \\
&= \int_0^{3\pi/2} -4\sin t\cos t + 2(1-\cos 2t) + 8\cos^2 t\sin t\, dt \\
&= -2\sin^2 t + 2t - \sin 2t - \frac{8}{3}\cos^3 t\Big|_0^{3\pi/2} \\
&= -2 + 3\pi + \frac{8}{3} = \boxed{3\pi + \frac{2}{3}}.
\end{aligned}$$

Interpreting Line Integrals

Given a vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ and a curve C parametrized by $\mathbf{r}(t)$ with $a \le t \le b$, we see that

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \\ &= \int_a^b \frac{\mathbf{F} \cdot \mathbf{r}'}{|\mathbf{r}'|} |\mathbf{r}'|\, dt \\ &= \int_C \operatorname{proj}_{\mathbf{T}}(\mathbf{F})\, ds, \end{aligned}$$

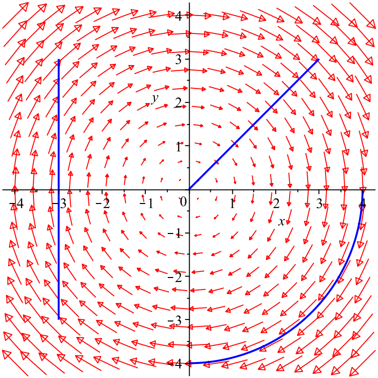
where $\mathbf{T}(t) = \mathbf{r}'(t)/|\mathbf{r}'(t)|$ is the unit tangent vector.

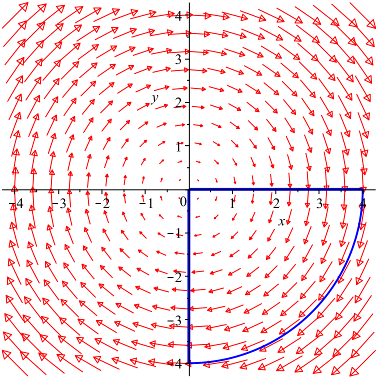
So $\int_C \mathbf{F} \cdot d\mathbf{r}$ is a net measurement of the tendency of $\mathbf{F}(x, y)$ to "agree with" motion along C.

Example

Example 2

For each curve C shown, choose an orientation and determine if $\int_C \mathbf{F} \cdot d\mathbf{r}$ is positive, negative or zero.





A Physical Interpretation

If $\mathbf{F}(x, y)$ represents a force at each point (e.g. gravity), and $\mathbf{r}(t)$ is the position of a particle at time t, subject to $\mathbf{F}$, then

$$\mathbf{F}(\mathbf{r}(t_i)) \cdot \mathbf{r}'(t_i)\Delta t_i \approx \mathbf{F}(\mathbf{r}(t_i)) \cdot \Delta \mathbf{r}_i,$$

which is the work done by $\mathbf{F}(\mathbf{r}(t_i))$ done through the displacement $\Delta \mathbf{r}_i$.

So at the level of Riemann sums we have

$$\sum_{i=1}^{n} \mathbf{F}(\mathbf{r}(t_i)) \cdot \mathbf{r}'(t_i)\Delta t_i \approx \sum_{i=1}^{n} \mathbf{F}(\mathbf{r}(t_i)) \cdot \Delta \mathbf{r}_i,$$

the RHS approximating the total work done by F as we move along $\mathbf{r}(t)$.

This approximation improves indefinitely as $\Delta t \to 0$, so that

$$\boxed{\int_C \mathbf{F} \cdot d\mathbf{r} = \text{ Total work done by } \mathbf{F} \text{ along } C.}$$

This suggests that we may interpret the general line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ as a measurement of the "energy" of C, relative to $\mathbf{F}$.

Put another way, if $\omega = P\,dx + Q\,dy$ is the 1-form corresponding to $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$, then the "evaluation" $E_\omega(C)$ represents the energy of C relative to ω.

A bit more specifically, suppose $\mathbf{F}(x, y)$ is a force field and $\mathbf{r}(t)$ represents the motion of a particle subject *only* to the force $\mathbf{F}$.

Newton's Second Law gives $\mathbf{F} = m\mathbf{r}''$ so that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b m\mathbf{r}''(t) \cdot \mathbf{r}'(t)\, dt = \frac{m}{2} \int_a^b \frac{d}{dt}(\mathbf{r}'(t) \cdot \mathbf{r}'(t))\, dt$$

$$= \frac{m}{2}\left(|\mathbf{r}'(b)|^2 - |\mathbf{r}'(a)|^2\right) = \frac{mv_f^2}{2} - \frac{mv_i^2}{2} = \Delta E_K$$

is the change in the kinetic energy of the particle between the initial and terminal points of C.

Conservative Vector Fields

Definition

We call the vector field $\mathbf{F}(x, y)$ *conservative* if $\mathbf{F} = \nabla f$ for some function $f(x, y)$. In this case we call f the *potential* of $\mathbf{F}$.

Warning. In physics, the function $-f$ is called the potential of $\mathbf{F}$. We will see why shortly.

Example. The vector field $\mathbf{F} = y\mathbf{i} + x\mathbf{j}$ is conservative since

$$\nabla(xy) = y\mathbf{i} + x\mathbf{j} = \mathbf{F}.$$

Its potential function is $f(x, y) = xy$.

Integration of Conservative Fields

Suppose that $\mathbf{F}(x, y)$ is conservative with potential $f(x, y)$.

Then on any curve C we have

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C f_x \, dx + f_y \, dy \\
&= \int_a^b \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \, dt \\
&= \int_a^b \frac{d}{dt} f(\mathbf{r}(t)) \, dt = f(\mathbf{r}(t)) \Big|_a^b \\
&= f(\text{end of } C) - f(\text{beg. of } C).
\end{aligned}$$

This is the *Fundamental Theorem of Calculus for Line Integrals*

FTOC for Line Integrals

Theorem 1 (FTOC for Line Integrals)

If $\mathbf{F}(x, y)$ is a conservative vector field with domain D and potential $f(x, y)$, then for any curve C in D from A to B we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A).$$

Consequently:

- $\int_C \mathbf{F} \cdot d\mathbf{r}$ *is* path independent; *it depends only on the* endpoints *of C.*
- $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ *for any* closed curve C *(when $A = B$).*

Exact 1-Forms

Given a function $f(x,y)$, its *differential* is

$$df = f_x\,dx + f_y\,dy.$$

If $\omega = P\,dx + Q\,dy$ and $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is the corresponding vector field, then

$$\mathbf{F} = \nabla f \quad \Leftrightarrow \quad \omega = df.$$

In this case we say that ω is *exact*, and the FTOC for line integrals becomes

$$\int_C df = f(\text{end of } C) - f(\text{beg. of } C).$$

Examples

Example 3

Evaluate $\int_C (x+2xy)\,dx + x^2\,dy$, where C is the curve consisting of the line segments from $(0,0)$ to $(2,1)$ to $(4,3)$ to $(5,0)$.

Solution. Notice that if $f(x,y) = x^2y + \frac{x^2}{2}$, then

$$df = (2xy + x)\,dx + x^2\,dy,$$

so that FTOC implies

$$\int_C (x+2xy)\,dx + x^2\,dy = \int_C df = f(5,0) - f(0,0) = \boxed{\frac{25}{2}}.$$

□

Example 4

Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = x\mathbf{i} + (y+2)\mathbf{j}$ and C is given by $\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t \rangle$, $0 \leq t \leq 2\pi$.

Solution. Notice that

$$\nabla \Bigg(\underbrace{\frac{x^2}{2} + \frac{y^2}{2} + 2y}_{f(x,y)} \Bigg) = x\mathbf{i} + (y+2)\mathbf{j} = \mathbf{F}.$$

Therefore, by FTOC,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(2\pi)) - f(\mathbf{r}(0)) = f(2\pi, 0) - f(0,0) = \boxed{2\pi^2}.$$

Remarks

Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ and $\omega = P\,dx + Q\,dy$.

Given $f(x, y)$, we can view ∇f and df as different versions of the *(total) derivative* of f.

Conversely, if $\mathbf{F}/\omega$ is conservative/exact, we can view the potential $f(x, y)$ as the antiderivative.

FTOC then says: if $\mathbf{F}/\omega$ has an antiderivative $f(x, y)$, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \omega = f(x, y)\Big|_{\text{beg.}}^{\text{end}},$$

which is an exact analogue of the single variable FTOC.

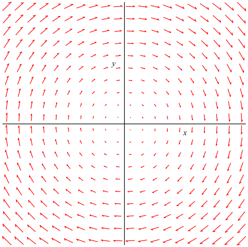
Warning!

Unlike the single variable FTOC, however, *not every vector field/1-form has an antiderivative*!

Example 5

Use the graph of $\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$ to show that it is not conservative.

Solution. Here's the graph of $\mathbf{F}$:



If C is any clockwise circle centered at the origin, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} > 0.$$

□

Conservation of Energy

Suppose that $\mathbf{F} = \nabla f$ is a conservative force field.

If a particle travels from A to B along a curve C subject only to $\mathbf{F}$, we have

$$E_K(B) - E_K(A) = \Delta E_K = \int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A),$$

or

$$E_K(B) - f(B) = E_K(A) - f(A).$$

If we define the *potential energy* to be $E_P = -f$, this becomes

$$E_K(B) + E_P(B) = E_K(A) + E_P(A),$$

which is the *law of conservation of energy*.

Forms and Fields in $\mathbb{R}^3$

A 1-form in 3 variables looks like

$$\omega = P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz = P\,dx + Q\,dy + R\,dz,$$

which corresponds to the 3D vector field

$$\mathbf{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}.$$

We say that $\mathbf{F}/\omega$ is conservative/exact provided

$$\mathbf{F} = \nabla f \quad \text{or } \omega = df = f_x\,dx + f_y\,dy + f_z\,dz.$$

The 3D analogue of the FTOC for line integrals also holds.

Example

Example 6

Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = (y+z)\mathbf{i} + (x+z)\mathbf{j} + (x+y)\mathbf{k}$ and C is the line segment from $(1,0,0)$ to $(3,4,2)$.

Solution. Observe that

$$\nabla(\underbrace{xy + xz + yz}_{f}) = (y+z)\mathbf{i} + (x+z)\mathbf{j} + (x+y)\mathbf{k}.$$

Therefore

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(3,4,2) - f(1,0,0) = \boxed{26}.$$