Conservative Vector Fields

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Calculus III

Introduction

Recall that a vector field $\mathbf{F}$ is called *conservative* provided that $\mathbf{F} = \nabla f$ for some function f.

Conservative fields are important because they obey the FTOC (for line integrals) and the law of conservation of energy.

We seek criteria that will help us identify conservative fields without specific reference to the underlying *potential function* f.

We will develop two such criteria, one in terms of line integrals, the other in terms of partial derivatives of the components of $\mathbf{F}$.

Characterizing Conservative Vector Fields

We begin with the following result.

Theorem 1

Let $\mathbf{F}$ *be a vector field (in* $\mathbb{R}^2$ *or* $\mathbb{R}^3$*) with domain* D*. The following are equivalent:*

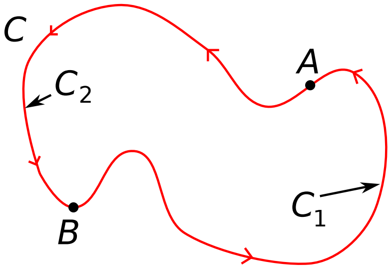
1. $\mathbf{F}$ *is conservative.*
2. $\int_C \mathbf{F} \cdot d\mathbf{r}$ *depends only on the endpoints of* C*.*
3. $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ *for every closed curve (loop) in* D*.*

Proof (Sketch). We have already seen that **1** implies both **2** and **3**.

We will show that **2** is equivalent to **3**, and that **2** implies **1**.

Suppose **2** is true, and let C be any closed curve in D.

Choose two points on C and split C into $C_1 + C_2$.



Since C_1 and $-C_2$ have the same endpoints, we have

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{-C_2} \mathbf{F} \cdot d\mathbf{r} = -\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

$$\Rightarrow \ 0 = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C} \mathbf{F} \cdot d\mathbf{r},$$

which proves **3**.

Conversely, assume **3**. Let C_1 and C_2 be two curves in D with the same endpoints.

Then $C_1 - C_2$ is a closed curve in D, and therefore

$$0 = \int_{C_1 - C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r} \ \Rightarrow \ \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r},$$

proving **2**.

It remains to show that **2** implies **1**, i.e. that path independence of $\int_C \mathbf{F} \cdot d\mathbf{r}$ implies **F** is conservative.

Fix a point $P_0 \in D$ and for $X \in D$ define

$$f(X) = \int_{P_0}^{X} \mathbf{F} \cdot d\mathbf{r},$$

where the integral is taken over any path in D from P_0 to X.

The function $f(X)$ is well-defined since $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path.

One can show that $\nabla f = \mathbf{F}$, so that **F** is conservative. $\square$

While Theorem 1 is theoretically quite nice, it does not provide a practical means for determining whether or not a vector field is conservative.

Suppose $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} = \nabla f = f_x\mathbf{i} + f_y\mathbf{j}$. Then

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(f_y) = f_{yx} = f_{xy} = \frac{\partial}{\partial y}(f_x) = \frac{\partial P}{\partial y},$$

by Clairaut's theorem. That is:

Theorem 2

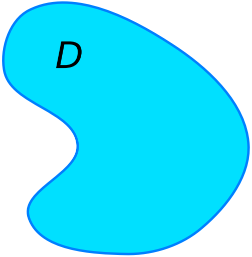
If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is a conservative C^1 vector field, then

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}.$$

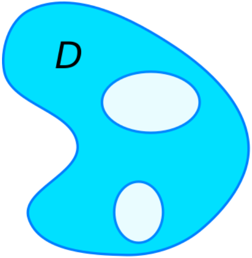
The converse of Theorem 2 is also true, provided we make an additional assumption about the domain D of $\mathbf{F}$.

Definition

We say that $D \subset \mathbb{R}^2$ is *simply connected* if every loop in D only encloses points in D.



Simply connected



Multiply connected

That is, D is simply connected if it has no "holes."

One can use Green's Theorem (which we will discuss next time) to prove:

Theorem 3

A vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ *with simply connected domain* $D \subset \mathbb{R}^2$ *is conservative if and only if*

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}.$$

Note that Theorem 3 applies, in particular, when $D = \mathbb{R}^2$, i.e. when $\mathbf{F}$ is defined everywhere.

Example

Example 1

Show that $\mathbf{F}(x, y) = (2xy + 2)\mathbf{i} + (x^2 - 2y)\mathbf{j}$ is conservative, and find its potential function.

Solution. $\mathbf{F}$ is defined everywhere and

$$\frac{\partial}{\partial x}(x^2 - 2y) = 2x = \frac{\partial}{\partial y}(2xy + 2),$$

so according to Theorem 3, $\mathbf{F}$ is conservative.

To find the potential we must solve the system

$$\begin{aligned} f_x &= 2xy + 2, \\ f_y &= x^2 - 2y. \end{aligned}$$

Integrating the first equation in x we have

$$f(x,y) = \int f_x(x,y)\,dx = \int 2xy + 2\,dx = x^2y + 2x + g(y),$$

where $g(y)$ is the "constant" of integration.

To find $g(y)$ we differentiate with respect to y and plug into the second equation:

$$\begin{aligned} x^2 + g'(y) = f_y(x,y) = x^2 - 2y \quad &\Rightarrow \quad g'(y) = -2y \\ &\Rightarrow \quad g(y) = -y^2 + C. \end{aligned}$$

Thus the potential is

$$\boxed{f(x,y) = x^2y + 2x - y^2}.$$

□

Warning

Warning. When applying Theorem 3, the hypothesis that the domain be simply connected is *essential*.

Consider the vector field

$$\mathbf{F}(x,y) = \frac{-y\mathbf{i} + x\mathbf{j}}{x^2+y^2},$$

whose domain is $\mathbb{R}^2 \setminus \{(0,0)\}$, which *is not simply connected*.

Let C be the unit circle, traversed counterclockwise:

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j},\ \ 0 \le t \le 2\pi.$$

Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (-\sin t\mathbf{i} + \cos t\mathbf{j}) \cdot (-\sin t\mathbf{i} + \cos t\mathbf{j})\, dt$$
$$= \int_0^{2\pi} \sin^2 t + \cos^2 t\, dt = \int_0^{2\pi} dt = 2\pi \neq 0,$$

which shows that **F** is *not* conservative.

However,

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{x^2+y^2-x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2},$$
$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{(x^2+y^2)(-1)+y(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}.$$

Closed 1-Forms

We can rephrase our work so far in terms of 1-forms as follows.

Definition

We define the *derivative* of a 1-form $\omega = P\,dx + Q\,dy$ to be

$$d\omega = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}.$$

We say that ω is *closed* if $d\omega = 0$.

If $\omega = df = f_x\,dx + f_y\,dy$ (ω is exact), then Clairaut's theorem implies

$$d\omega = d^2 f = f_{yx} - f_{xy} = 0.$$

That is:

Theorem 4

Every exact 1-form (in two variables) is closed.

We can restate Theorem 3 as follows.

Theorem 5 (Theorem 3 for 1-Forms)

Every closed 1-form with a simply connected domain is exact.

However, as the example

$$\omega = \frac{-y\,dx + x\,dy}{x^2 + y^2}$$

shows:

$$\boxed{\text{not every closed form (in general) is exact}}.$$

The precise extent to which there exist non-exact closed 1-forms on a domain $D \subset \mathbb{R}^2$ is measured by its *DeRham cohomology*.

Conservative Fields in $\mathbb{R}^3$

What about conservative fields in three variables?

Suppose $\mathbf{F}(x, y, z) = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is conservative.

Then there is a function $f(x, y, z)$ so that

$$P = f_x, \;\; Q = f_y, \;\; R = f_z.$$

Clairaut's theorem then tells us that

$$\frac{\partial P}{\partial y} = f_{xy} = f_{yx} = \frac{\partial Q}{\partial x} \;\; \Rightarrow \;\; \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0.$$

We also have

$$\frac{\partial P}{\partial z} = f_{xz} = f_{zx} = \frac{\partial R}{\partial x} \quad \Rightarrow \quad \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} = 0,$$

$$\frac{\partial Q}{\partial z} = f_{yz} = f_{zy} = \frac{\partial R}{\partial y} \quad \Rightarrow \quad \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} = 0.$$

We can simultaneously represent these three conditions with the single vector equation

$$\mathbf{0} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = \nabla \times \mathbf{F},$$

where $\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$.

The Curl

Definition

The *curl* of $\mathbf{F}(x, y, z)$ is the vector field

$$\text{curl}(\mathbf{F}) = \nabla \times \mathbf{F}.$$

Our computations above prove:

Theorem 6

If $\mathbf{F}(x, y, z)$ *is conservative, then* $\text{curl}(\mathbf{F}) = \mathbf{0}$.

As with 2D vector fields, this result has a partial converse:

Theorem 7

If the domain of $\mathbf{F}(x, y, z)$ *is free from "holes" (e.g. is all of* $\mathbb{R}^3$*) and* $\text{curl}(\mathbf{F}) = \mathbf{0}$, *then* $\mathbf{F}$ *is conservative.*

Examples

Example 2

Show that the vector field

$$\mathbf{F}(x, y, z) = (3x + z)\mathbf{i} + (2x - y)\mathbf{j} + (z - y)\mathbf{k}$$

is *not* conservative.

Solution. We simply compute

$$\operatorname{curl}(\mathbf{F}) = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3x + z & 2x - y & z - y \end{vmatrix}$$

$$= (-1 - 0)\,\mathbf{i} + (1 - 0)\,\mathbf{j} + (2 - 0)\,\mathbf{k} \neq \mathbf{0}.$$

Therefore $\mathbf{F}$ cannot be conservative. □

Example 3

Show that the vector field

$$\mathbf{F}(x,y,z) = y(z+2)\mathbf{i} + (xz+2x+4)\mathbf{j} + (xy+3)\mathbf{k}$$

is conservative and find its potential.

Solution. First we compute

$$\begin{aligned}\text{curl}(\mathbf{F}) = \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz+2y & xz+2x+4 & xy+3 \end{vmatrix} \\ &= (x-x)\mathbf{i} - (y-y)\mathbf{j} + (z+2-(z+2))\mathbf{k} = \mathbf{0}.\end{aligned}$$

Since $\mathbf{F}$ is defined everywhere, this proves it is conservative.

To find the potential we must solve the system

$$f_x = yz + 2y, \;\; f_y = xz + 2x + 4, \;\; f_z = xy + 3. \qquad (1)$$

Integrating the first with respect to x gives

$$f(x, y, z) = \int f_x(x, y, z)\, dx = \int yz + 2y\, dx = xyz + 2xy + g(y, z),$$

where $g(y, z)$ is the "constant" of integration.

To find g, we differentiate in y and compare to (1):

$$xz + 2x + 4 = f_y = xz + 2x + g_y \;\; \Rightarrow \;\; g_y(y, z) = 4.$$

Now integrate in y:

$$g(y,z) = \int g_y(y,z)\,dy = \int 4\,dy = 4y + h(z),$$

where $h(z)$ is another "constant."

So $f(x,y,z) = xyz + 2xy + g(y,z) = xyz + 2xy + 4y + h(z)$.

Finally, compute f_z and compare:

$$xy + 3 = f_z = xy + h'(z) \;\Rightarrow\; h'(z) = 3 \;\Rightarrow\; h(z) = 3z + C.$$

Thus, any function of the form

$$\boxed{f(x,y,z) = xyz + 2xy + 4y + 3z + C}$$

serves as a potential for $\mathbf{F}$. □