Green's Theorem

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Calculus III

Introduction

Recall that if $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is a conservative vector field and C is a closed curve (loop), then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

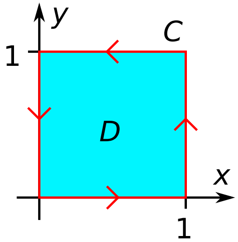
Furthermore, if the domain D of $\mathbf{F}$ is simply connected (has no "holes"), then $\mathbf{F}$ is conservative if and only if

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0.$$

Green's theorem provides an explanation of this phenomenon by giving a direct relationship between the integrals of $\mathbf{F}$ and $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$, for arbitrary $\mathbf{F}$.

Motivating Example

Consider $\int_C P\,dx + Q\,dy$, where C is the boundary of the unit square $[0,1] \times [0,1]$, travelled counterclockwise:



Let L, R, T, B denote the left, right, top and bottom edges of C, respectively.

On R and L we have $dx = 0$ so that

$$\int_R P\,dx + Q\,dy = \int_R Q\,dy = \int_0^1 Q(1,y)\,dy,$$
$$\int_L P\,dx + Q\,dy = \int_L Q\,dy = -\int_0^1 Q(0,y)\,dy.$$

Thus

$$\int_{L+R} P\,dx + Q\,dy = \int_0^1 Q(1,y) - Q(0,y)\,dy = \int_0^1 Q(x,y)\Big|_{x=0}^{x=1}\,dy$$
$$= \int_0^1 \int_0^1 Q_x(x,y)\,dx\,dy = \iint_D \frac{\partial Q}{\partial x}\,dA.$$

In an entirely similar fashion one can also show that

$$\begin{aligned}\int_{B+T} P\,dx + Q\,dy &= \int_{B+T} P\,dx = -\int_0^1 P(x,1) - P(x,0)\,dx \\ &= -\int_0^1 P(x,y)\Big|_{y=0}^{y=1} dx = -\int_0^1\int_0^1 P_y(x,y)\,dy\,dx \\ &= -\iint_D \frac{\partial P}{\partial y}\,dA.\end{aligned}$$

Therefore

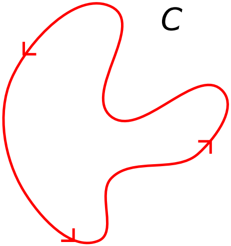
$$\int_C P\,dx + Q\,dy = \int_{B+R+T+L} P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA.$$

This is a particular instance of *Green's Theorem*.

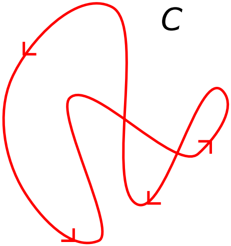
Preliminary Definitions

Definition

A curve C is called *closed* if its initial and terminal points are the same (i.e. C is a loop). We say that a closed curve is *simple* if it never crosses through itself.



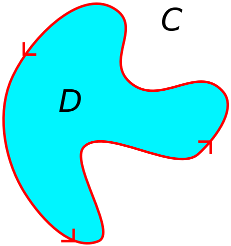
Simple, closed



Closed, not simple

Definition

Let C be a simple closed curve enclosing a region $D \subset \mathbb{R}^2$. We say that C is *positively oriented* if D is always to the left as you travel along C.



Positive orientation

Finally, recall that a region $D \subset \mathbb{R}^2$ is called *simply connected* if it has no "holes."

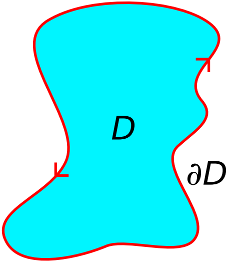
Green's Theorem

We can now state our main result of the day.

Theorem 1 (Green's Theorem)

Let $D \subset \mathbb{R}^2$ be a simply connected region with positively oriented simple closed boundary curve ∂D. If $P, Q \in C^1(D)$, then

$$\int_{\partial D} P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA.$$



The setup for Green's Theorem.

Remark. Green's Theorem generalizes the fact that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for a conservative field $\mathbf{F}$ and a closed curve C.

Remark

Recall that if $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$,then

$$\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix}.$$

If we view $\mathbf{F}(x, y) = P\mathbf{i} + Q\mathbf{j}$ as having zero z-component, then this becomes

$$\operatorname{curl}(\mathbf{F}) = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}.$$

If we drop the $\mathbf{k}$, we can restate Green's theorem in vector form as

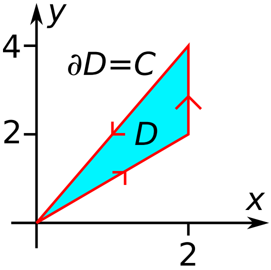
$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \iint_D \operatorname{curl}(\mathbf{F})\, dA.$$

Examples

Example 1

Evaluate $\int_C xy^2\,dx + 2x^2y\,dy$, where C is the triangle with vertices $(0,0)$, $(2,2)$, $(2,4)$, oriented positively.

Solution.



We apply Green's theorem.

The region D is Type I, with "bottom" $y = x$ and "top" $y = 2x$, for $0 \leq x \leq 2$.

Thus

$$\int_C xy^2\,dx + 2x^2y\,dy = \int_0^2 \int_x^{2x} \frac{\partial}{\partial x}(2x^2y) - \frac{\partial}{\partial y}(xy^2)\,dy\,dx$$

$$= \int_0^2 \int_x^{2x} 4xy - 2xy\,dy\,dx = \int_0^2 \int_x^{2x} 2xy\,dy\,dx$$

$$= \int_0^2 xy^2\Big|_{y=x}^{y=2x}\,dx = \int_0^2 3x^3\,dx$$

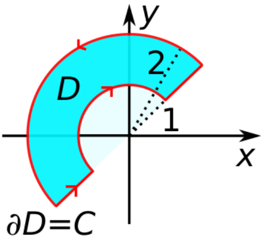
$$= \frac{3x^4}{4}\Big|_0^2 = \boxed{12}.$$

□

Example 2

Evaluate $\int_C xe^{-2x}\,dx + (x^4 + 2x^2y^2)\,dy$, where C is the boundary of the semi-annulus shown below.

Solution.



We apply Green's theorem.

Note that the region D enclosed by C has polar description $1 \leq r \leq 2$, $\pi/4 \leq \theta \leq 5\pi/4$.

Therefore

$$\int_C xe^{-2x}\,dx+(x^4+2x^2y^2)\,dy = \iint_D \frac{\partial}{\partial x}(x^4+2x^2y^2) - \frac{\partial}{\partial y}(xe^{-2x})\,dA$$

$$= \iint_D 4x^3+4xy^2\,dA = \int_{\pi/4}^{5\pi/4}\int_1^2 4r^4\cos\theta\,dr\,d\theta$$

$$= 4\int_{\pi/4}^{5\pi/4}\cos\theta\,d\theta \times \int_1^2 r^4\,dr = 4\left(\sin\theta\Big|_{\pi/4}^{5\pi/4}\right)\left(\frac{r^5}{5}\Big|_1^2\right)$$

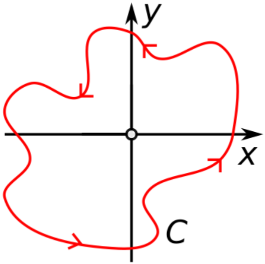
$$= 4\left(-\sqrt{2}\right)\left(\frac{2^5-1}{5}\right) = \boxed{-\frac{124\sqrt{2}}{5}}.$$

□

Example 3

Evaluate $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \dfrac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2}$ and C is any positively oriented simple closed curve enclosing the origin.

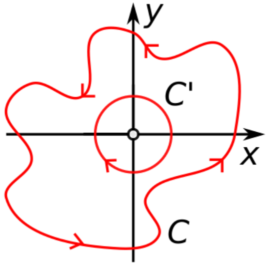
Solution. Although the region enclosed by C is simply connected, the domain of $\mathbf{F}$ does not include $(0, 0)$ so Green's theorem does not apply.



Example 4

Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \dfrac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2}$ and C is any positively oriented simple closed curve enclosing the origin.

Solution. Although the region enclosed by C is simply connected, the domain of $\mathbf{F}$ does not include $(0, 0)$ so Green's theorem does not apply.



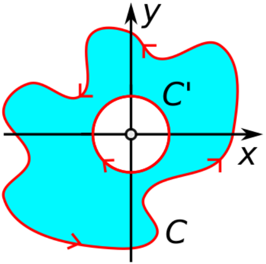
Let C' denote a small circle of radius a centered at the origin and enclosed by C.

Introduce line segments along the x-axis and split the region between C and C' in two.

Example 5

Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \dfrac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2}$ and C is any positively oriented simple closed curve enclosing the origin.

Solution. Although the region enclosed by C is simply connected, the domain of $\mathbf{F}$ does not include $(0, 0)$ so Green's theorem does not apply.



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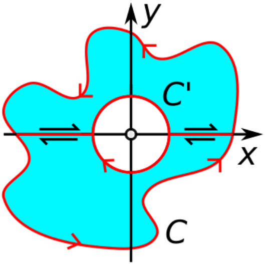
Introduce line segments along the x-axis and split the region between C and C' in two.

Example 6

Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \dfrac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2}$ and C is any positively oriented simple closed curve enclosing the origin.

Solution. Although the region enclosed by C is simply connected, the domain of $\mathbf{F}$ does not include $(0, 0)$ so Green's theorem does not apply.



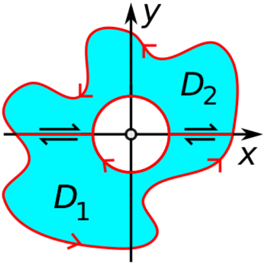
Let C' denote a small circle of radius a centered at the origin and enclosed by C.

Introduce line segments along the x-axis and split the region between C and C' in two.

Example 7

Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \dfrac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2}$ and C is any positively oriented simple closed curve enclosing the origin.

Solution. Although the region enclosed by C is simply connected, the domain of $\mathbf{F}$ does not include $(0,0)$ so Green's theorem does not apply.



Let C' denote a small circle of radius a centered at the origin and enclosed by C.

Introduce line segments along the x-axis and split the region between C and C' in two.

Because the integrals along the line segments cancel out we have

$$\int_{C+C'} \mathbf{F} \cdot d\mathbf{r} = \int_{\partial D_1 + \partial D_2} \mathbf{F} \cdot d\mathbf{r} = \int_{\partial D_1} \mathbf{F} \cdot d\mathbf{r} + \int_{\partial D_2} \mathbf{F} \cdot d\mathbf{r}.$$

We *can* apply Green's theorem on D_1 and D_2. Since $\text{curl}(\mathbf{F}) = 0$ (previous lecture) we have

$$\int_{\partial D_1} \mathbf{F} \cdot d\mathbf{r} + \int_{\partial D_2} \mathbf{F} \cdot d\mathbf{r} = \iint_{D_1} 0\, dA + \iint_{D_2} 0\, dA = 0.$$

It now follows that

$$\int_C \mathbf{F} \cdot \mathbf{r} = \int_{-C'} \mathbf{F} \cdot \mathbf{r}.$$

We parametrize $-C'$ in the usual way:

$$\mathbf{r}(t) = \langle a\cos t, a\sin t\rangle \;\Rightarrow\; \mathbf{r}'(t) = \langle -a\sin t, a\cos t\rangle,$$

with $0 \le t \le 2\pi$. Thus

$$\int_{-C'} \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \frac{-a\sin t\,\mathbf{i} + a\cos t\,\mathbf{j}}{a^2\cos^2 t + a^2\sin^2 t}\cdot\langle -a\sin t, a\cos t\rangle\,dt$$

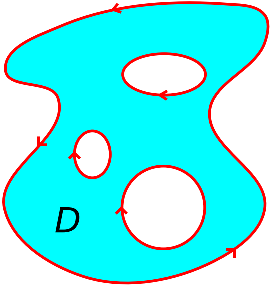
$$= \int_0^{2\pi} \frac{a^2}{a^2}\,dt = \boxed{2\pi}.$$

□

Remarks

Using a subdivision argument as in the preceding example, one can show that Green's theorem applies to a multiply connected region D provided:

1. The boundary ∂D consists of multiple simple closed curves.
2. Each piece of ∂D is positively oriented *relative to* D.



$$\int_{\partial D} P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$$

for $P, Q \in C^1(D)$.

Remarks (Cont.)

Suppose D is a region to which Green's theorem applies. Then

$$\text{Area}(D) = \iint_D dA = \frac{1}{2}\iint_D \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\,dA$$
$$= \frac{1}{2}\int_{\partial D} -y\,dx + x\,dy.$$

That is, we can compute the area of D by integrating $\omega = \frac{1}{2}(-y\,dx + x\,dy)$ around ∂D.

This observation provides the theoretical basis behind what's known as a *planimeter*, which can compute the area of a plane region by tracing its boundary.

Example 8

Suppose we are given n vertices of a polygon P: (x_1, y_1), (x_2, y_2), ..., (x_n, y_n). Use Green's theorem to derive a formula for the area of P only in terms of the coordinates of its vertices.

Solution. One can show (HW) that if L is the line segment from (a, b) to (c, d), then

$$\int_L -y\,dx + x\,dy = ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

Assuming that ∂P is oriented positively, our area formula above tells us that

$$\text{Area}(P) = \frac{1}{2}\int_{\partial P} -y\,dx + x\,dy.$$

Since the boundary of P consists of the line segments between the (x_i, y_i) we find that

$$\frac{1}{2}\int_{\partial P} -y\,dx + x\,dy = \boxed{\frac{1}{2}\sum_{i=1}^{n}\begin{vmatrix} x_i & y_i \\ x_{i+1} & y_{i+1}\end{vmatrix}},$$

where we define $(x_{n+1}, y_{n+1}) = (x_1, y_1)$. □

Example 9

Compute the area of the polygon with vertices $(0,0)$, $(4,3)$, $(3,4)$, $(-1,3)$, $(-2,1)$ and $(1,2)$.

Solution. We use the preceding example:

$$\text{Area}(P) = \frac{1}{2}\left(\begin{vmatrix} 0 & 0 \\ 4 & 3\end{vmatrix} + \begin{vmatrix} 4 & 3 \\ 3 & 4\end{vmatrix} + \begin{vmatrix} 3 & 4 \\ -1 & 3\end{vmatrix} + \begin{vmatrix} -1 & 3 \\ -2 & 1\end{vmatrix} + \begin{vmatrix} -2 & 1 \\ 1 & 2\end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 0 & 0\end{vmatrix}\right)$$

$$= \frac{1}{2}((0-0)+(16-9)+(9+4)+(-1+6)+(-4-1)+(0-0))$$

$$= \frac{1}{2}(7+13+5-5) = \boxed{10}.$$

$\square$

Have a great Thanksgiving break!!!