Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$

Ryan C. Daileda

Trinity University

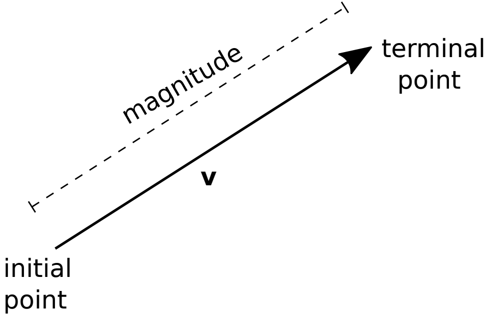
Calculus III

Introduction

- *Vectors* represent quantities that have both size (*magnitude*) and direction.
- Examples of vector quantities: displacement, velocity, acceleration, force, electric/magnetic fields, etc.
- In the context of vectors, real numbers are referred to as *scalars*.
- Examples of scalar quantities: speed, mass, temperature, energy, etc.

Representing Vectors

A vector in $\mathbb{R}^2$ or $\mathbb{R}^3$ can be visualized as an arrow:



Remarks

- We use boldface type for variables that represent vectors, e.g. $\mathbf{u}, \mathbf{v}, \mathbf{w}$.
- Alternate terminology:

  magnitude = *length*

  terminal point = *tip*

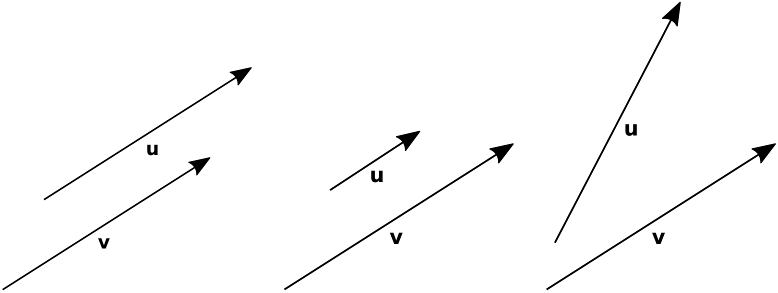
  initial point = *tail*

- The magnitude of a vector $\mathbf{v}$ is denoted $|\mathbf{v}|$.
- There's no standard way to denote the direction of a vector.

Vector Equality

Two vectors $\mathbf{u}$ and $\mathbf{v}$ are considered *equal* (i.e. $\mathbf{u} = \mathbf{v}$) whenever they have *the same magnitude and direction*.

Equality of vectors *does not depend on their location in space*.



$\mathbf{u} = \mathbf{v}$ $\mathbf{u} \neq \mathbf{v}$ $\mathbf{u} \neq \mathbf{v}$

The Zero Vector

As with numbers, it is useful to have a vector with no "content."

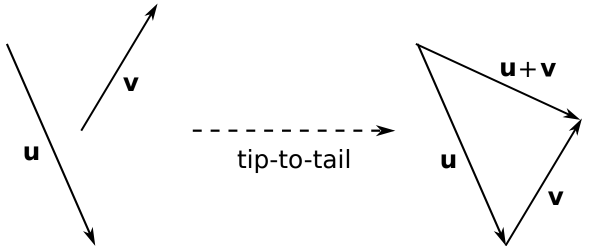
The *zero vector*, denoted $\mathbf{0}$, has *no magnitude and no direction*. It is simply a point:

$$\mathbf{0}$$

The tip and tail of $\mathbf{0}$ coincide, so $|\mathbf{0}| = 0$.

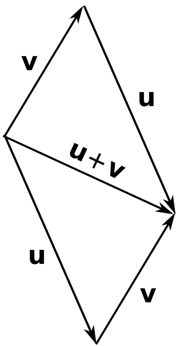
Vector Arithmetic

Vector Addition: Two vectors can be *added* by positioning them "tip-to-tail," and then inserting a vector from the tail of the first to the tip of the second.



The following diagram shows that vector addition is *commutative*:

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.$$



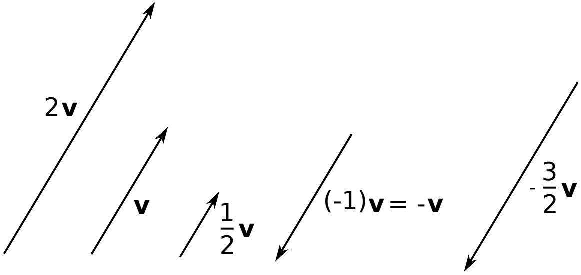
It is also geometrically clear that the zero vector “acts like zero”:

$$\mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v} = \mathbf{v}.$$

Scalar Multiplication: If $\mathbf{v}$ is a vector and c is a scalar, then the *scalar multiple* $c\mathbf{v}$ is the vector with the following magnitude and direction:

- $|c\mathbf{v}| = |c| \cdot |\mathbf{v}|$
- $\text{direction of } c\mathbf{v} = \begin{cases} \text{direction of } \mathbf{v} & \text{if } c > 0, \\ \text{opp. dir. of } \mathbf{v} & \text{if } c < 0. \end{cases}$

That is, $c\mathbf{v}$ "scales" $\mathbf{v}$ by a factor of c (hence the term "scalar").



Notice that $|0\mathbf{v}| = |0| \cdot |\mathbf{v}| = 0 \cdot |\mathbf{v}| = 0$, which means that

$$0\mathbf{v} = \mathbf{0}.$$

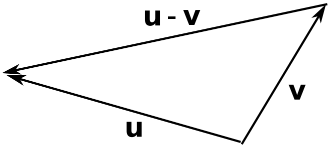
So again zero "acts like zero."

Vector Subtraction:

We combine addition and scalar multiplication and define

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}).$$

Because $-\mathbf{v}$ interchanges the tip and tail of $\mathbf{v}$, we have the following diagram:



That is, we subtract vectors by placing them "tail-to-tail."

Notice that, in particular, this means $\mathbf{v} - \mathbf{v} = \mathbf{0}$.

Remarks

- Addition and scalar multiplication are geometrically defined operations on vectors.
- We call these operations "addition" and "multiplication" because they obey many of the same laws that ordinary addition and multiplication do.
- For example, addition is associative:

  $$(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w}).$$

  And the distributive law holds:

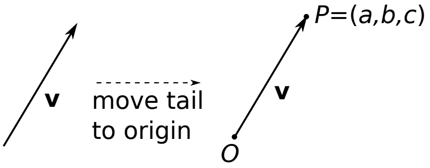
  $$c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}.$$

- See the box on page 802 of the text for a more complete list of the algebraic properties of these operations.

Vector Components

We now consider vectors from the analytic point of view.

Given a vector **v** (in $\mathbb{R}^2$ or $\mathbb{R}^3$) we position it with its tail at the origin:



In this configuration, the coordinates of the tip of **v** (the point P) are the *components* of **v**.

We write $\mathbf{v} = \langle a, b, c \rangle$.

Remarks

- Both points and vectors now have "coordinates." The difference between the two is primarily how we interpret them:
  - * The coordinates of a point tell you its location.
  - * The components of a vector tell you where the tip is, relative to the tail.
- Vector components provide a symbolic means of talking about magnitude and direction.
- We will focus on components in $\mathbb{R}^3$, but everything that follows holds for $\mathbb{R}^2$ as well, if one simply "forgets" the third component.

Vector Arithmetic with Components

The geometric operations of vector addition and scalar multiplication are easily represented using components.

If $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, then:

1. $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$
2. $c\mathbf{v} = \langle cv_1, cv_2, cv_3 \rangle$
3. $|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$

Thus, analytically we add and scale vectors "component-wise."

Examples

Example 1

If $\mathbf{a} = \langle 1, 2 - 3\rangle$ and $\mathbf{b} = \langle -2, -1, 5\rangle$, find $\mathbf{a}+\mathbf{b}$, $2\mathbf{a}+3\mathbf{b}$ and $|\mathbf{a}-\mathbf{b}|$.

Solution. We have

$$\mathbf{a}+\mathbf{b} = \langle 1+(-2), 2+(-1), -3+5\rangle = \boxed{\langle -1, 1, 2\rangle.}$$

Likewise

$$2\mathbf{a}+3\mathbf{b} = \langle 2, 4, -6\rangle + \langle -6, -3, 15\rangle = \boxed{\langle -4, 1, 9\rangle.}$$

Finally

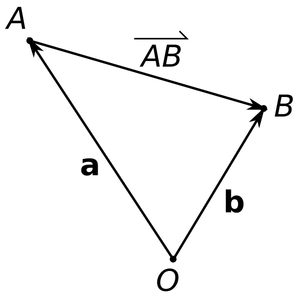
$$|\mathbf{a}-\mathbf{b}| = \left|\langle 1-(-2), 2-(-1), -3-5\rangle\right| = \left|\langle 3, 3, -8\rangle\right|$$

$$= \sqrt{3^2 + 3^2 + (-8)^2} = \boxed{\sqrt{82}.}$$

Example 2

Find the components of the vector that points from $A = (a_1, a_2, a_3)$ to $B = (b_1, b_2, b_3)$.

Solution. We introduce the origin O and consider the following diagram:



Because **a** begins at the origin and ends at A, its components are the coordinates of A:

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle.$$

Likewise we have

$$\mathbf{b} = \langle b_1, b_2, b_3 \rangle.$$

Because **a** and **b** are tail-to-tail, $\overrightarrow{AB}$ must be their difference:

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \boxed{\langle b_1 - a_1, b_2 - a_2, b_3 - a_3 \rangle.}$$

Remark: To find $\overrightarrow{AB}$ we treat A and B as vectors and *subtract the beginning from the end.*

Example 3

Find a *unit vector* (vector of length 1) that points in the direction of $\mathbf{v} = \langle -1, 3, 4 \rangle$.

Solution. Notice that the vector $\frac{1}{|\mathbf{v}|}\mathbf{v}$ has magnitude

$$\left| \frac{1}{|\mathbf{v}|}\mathbf{v} \right| = \frac{1}{|\mathbf{v}|} \cdot |\mathbf{v}| = 1,$$

and the same direction as $\mathbf{v}$ since $1/|\mathbf{v}| > 0$.

So we simply need to "divide" $\mathbf{v}$ by $|\mathbf{v}|$ to obtain the vector we want:

$$\mathbf{u} = \frac{1}{\sqrt{(-1)^2 + 3^2 + 4^2}} \langle -1, 3, 4 \rangle = \boxed{\left\langle \frac{-1}{\sqrt{26}}, \frac{3}{\sqrt{26}}, \frac{4}{\sqrt{26}} \right\rangle}.$$

The Standard Basis

The *standard basis vectors* are

$$\mathbf{i} = \langle 1, 0, 0 \rangle, \ \ \mathbf{j} = \langle 0, 1, 0 \rangle, \ \ \mathbf{k} = \langle 0, 0, 1 \rangle.$$

They represent unit displacements along the coordinate axes.

Every vector can be expressed in terms of **i**, **j** and **k**:

$$\begin{aligned} \langle a, b, c \rangle &= \langle a, 0, 0 \rangle + \langle 0, b, 0 \rangle + \langle 0, 0, c \rangle \\ &= a\langle 1, 0, 0 \rangle + b\langle 0, 1, 0 \rangle + c\langle 0, 0, 1 \rangle \\ &= a\mathbf{i} + b\mathbf{j} + c\mathbf{k}. \end{aligned}$$

This alternate notation will be useful when we discuss the *cross product* of vectors.

Examples

1. $\langle 1, 2, 3\rangle = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

2. $\langle -1, 0, 2\rangle = -\mathbf{i} + 2\mathbf{k}$

3. $\mathbf{k} - 3\mathbf{j} = \langle 0, -3, 1\rangle$

4. In $\mathbb{R}^2$ we only need $\mathbf{i} = \langle 1, 0\rangle$ and $\mathbf{j} = \langle 0, 1\rangle$:

$$\langle 3, -2\rangle = 3\mathbf{i} - 2\mathbf{j}.$$