The Dot Product

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Calculus III

Introduction

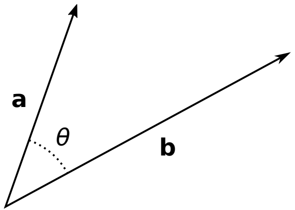
- Today we will introduce the first of two ways to "multiply" vectors together: the *dot product*.
- The dot product is an algebraically useful device that we will use to construct the equations of planes (among other things).
- Although both analytically and geometrically simple enough, the dot product fails to have a concrete "meaning," making it somewhat subtle to understand.

The Dot Product

Given vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, their *dot product* is given by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$= |\mathbf{a}| \cdot |\mathbf{b}| \cdot \cos\theta$$



The equality of the analytic (first) and geometric (second) expressions follows from the Law of Cosines.

Remarks

- We *always* measure θ when **a** and **b** are tail-to-tail.
- We always use the "inner" angle between **a** and **b**, that is

$$0 \leq \theta \leq \pi.$$

- In $\mathbb{R}^2$ we simply "forget" the third component: if $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$, then

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2.$$

- $\mathbf{a} \cdot \mathbf{b}$ is a *scalar*.

- As an operation $\cdot$ "acts like" multiplication, e.g.

$$\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}, \qquad \mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c},$$
$$\mathbf{a}\cdot\mathbf{0} = 0.$$

  See Section 12.3 of the textbook for more.

- Notice that if $\mathbf{a} = \langle a_1, a_2, a_3\rangle$, then

$$\boxed{\mathbf{a}\cdot\mathbf{a} = a_1^2 + a_2^2 + a_3^2 = |\mathbf{a}|^2.}$$

- The dot product gives an analytic expression for the angle between two vectors $\mathbf{a}$ and $\mathbf{b}$:

$$\boxed{\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\cdot|\mathbf{b}|}}$$

Examples

Example 1

Find the angle between $\mathbf{a} = \langle 1, 2, 3 \rangle$ and $\mathbf{b} = \langle 0, -2, 1 \rangle$.

Solution. We use the preceding formula. First we have

$$\mathbf{a} \cdot \mathbf{b} = 1 \cdot 0 + 2 \cdot (-2) + 3 \cdot 1 = -1,$$

$$|\mathbf{a}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14},$$

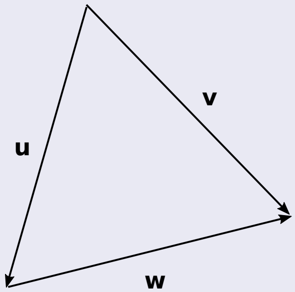
$$|\mathbf{b}| = \sqrt{0^2 + (-2)^2 + 1^2} = \sqrt{5}.$$

Thus

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| \cdot |\mathbf{b}|} = \frac{-1}{\sqrt{14} \cdot \sqrt{5}} \Rightarrow \boxed{\theta = \arccos\left(\frac{-1}{\sqrt{70}}\right) \approx 96.9^\circ}$$

Example 2

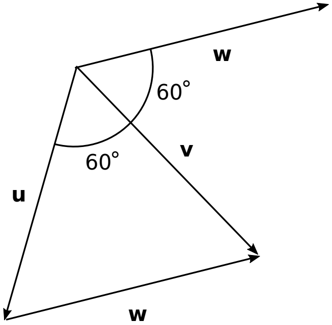
The vectors **u**, **v** and **w** form an equilateral triangle, as shown below. If $|\mathbf{u}| = 2$, find $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{u} \cdot \mathbf{w}$.



Solution. The angle between **u** *and* **v** *is* 60°*, so*

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| \cdot |\mathbf{v}| \cdot \cos 60^\circ = 2 \cdot 2 \cdot \frac{1}{2} = \boxed{2.}$$

To find the angle between **u** and **w** we must place them tail-to-tail:



We find that

$$\mathbf{u} \cdot \mathbf{w} = |\mathbf{u}| \cdot |\mathbf{v}| \cdot \cos 120^\circ = 2 \cdot 2 \cdot \left(-\frac{1}{2}\right) = \boxed{-2.}$$

Interpreting the Dot Product

Because $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \cdot |\mathbf{b}| \cdot \cos\theta$, $|\mathbf{a}| \geq 0$ and $|\mathbf{b}| \geq 0$, for nonzero vectors we have

$$\text{sign of } \mathbf{a} \cdot \mathbf{b} = \text{ sign of } \cos\theta = \begin{cases} +1 & \text{if } 0 \leq \theta < \pi/2, \\ -1 & \text{if } \pi/2 < \theta \leq \pi, \\ 0 & \text{if } \theta = \pi/2. \end{cases}$$

So the sign of $\mathbf{a} \cdot \mathbf{b}$ measures whether $(+1)$ or not (-1) the angle between $\mathbf{a}$ and $\mathbf{b}$ is acute.

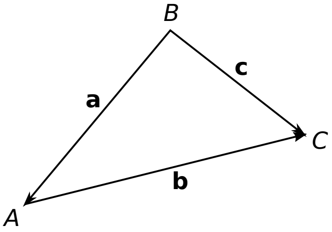
When the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\pi/2$, we say $\mathbf{a}$ and $\mathbf{b}$ are *orthogonal*. Thus:

$$\boxed{\mathbf{a} \text{ and } \mathbf{b} \text{ are orthogonal} \iff \mathbf{a} \cdot \mathbf{b} = 0.}$$

Example 3

Show that the points $A = (4, 0, 5)$, $B = (1, 1, 1)$ and $C = (3, 3, 0)$ are the vertices of a right triangle.

Solution. We (roughly) sketch the points and introduce vectors for the edges of the triangle:



The question now is: are any of $\mathbf{a} \cdot \mathbf{b}$, $\mathbf{a} \cdot \mathbf{c}$, $\mathbf{b} \cdot \mathbf{c}$ zero?

We have

$$\begin{aligned}
\mathbf{a} &= \overrightarrow{BA} = \langle 4-1, 0-1, 5-1\rangle = \langle 3, -1, 4\rangle,\\
\mathbf{b} &= \overrightarrow{AC} = \langle 3-4, 3-0, 0-5\rangle = \langle -1, 3, -5\rangle,\\
\mathbf{c} &= \overrightarrow{BC} = \langle 3-1, 3-1, 0-1\rangle = \langle 2, 2, -1\rangle.
\end{aligned}$$

Since

$$\mathbf{a}\cdot\mathbf{b} = 3\cdot(-1) + (-1)\cdot 3 + 4\cdot(-5) = -26 \neq 0,$$

$\angle BAC \neq 90^\circ$ (it's acute). But

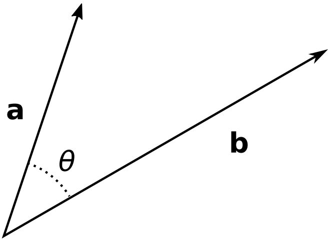
$$\mathbf{a}\cdot\mathbf{c} = 3\cdot 2 + (-1)\cdot 2 + 4(-1) = 0,$$

so that $\angle BCA = 90^\circ$, and hence $\triangle ABC$ is right.

Projections

The dot product is intimately connected with the notion of *orthogonal projection*.

Suppose we are given vectors **a** and **b**:



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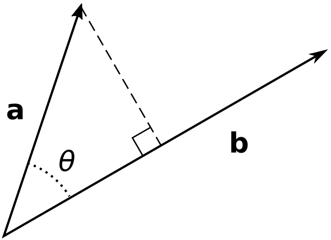

Introduce a perpendicular, as shown.

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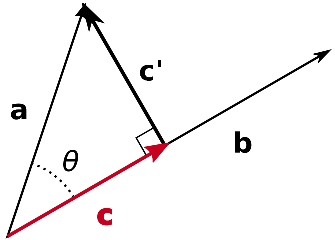
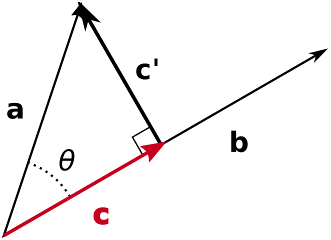


Introduce a perpendicular, as shown; then the vectors **c** and **c**′.

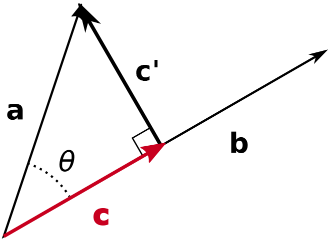


Notice that

$$\mathbf{a} = \mathbf{c} + \mathbf{c}'$$

and that

$\mathbf{c}$ is parallel to $\mathbf{b}$,
$\mathbf{c}'$ is orthogonal to $\mathbf{b}$.



Let's compute **c**. First of all

$$\cos\theta = \frac{|\mathbf{c}|}{|\mathbf{a}|} \quad \Rightarrow \quad |\mathbf{c}| = |\mathbf{a}|\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}$$

Next we need a unit vector with the direction of **b**: $\mathbf{u} = \frac{\mathbf{b}}{|\mathbf{b}|}$ works.

It now follows that

$$\mathbf{c} = |\mathbf{c}|\mathbf{u} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}\right)\frac{\mathbf{b}}{|\mathbf{b}|} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\right)\mathbf{b}.$$

Based on these computations we define:

Definition

The *scalar projection of* **a** *on* **b** is

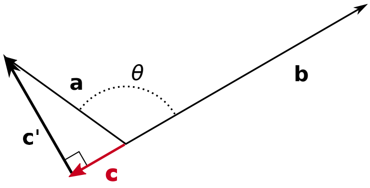
$$\mathrm{comp}_{\mathbf{b}}(\mathbf{a}) = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}.$$

The *vector projection of* **a** *on* **b** is

$$\mathrm{proj}_{\mathbf{b}}(\mathbf{a}) = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\right)\mathbf{b}.$$

Remarks

- Our derivation assumed that $\theta < \pi/2$, but the formulae above are valid for all values of θ.
- When $\theta > \pi/2$:
  - * $\text{comp}_{\mathbf{b}}(\mathbf{a}) < 0$;
  - * $\text{proj}_{\mathbf{b}}(\mathbf{a})$ points opposite $\mathbf{b}$.



- $\text{comp}_{\mathbf{b}}(\mathbf{a})$ is the "signed" length of $\text{proj}_{\mathbf{b}}(\mathbf{a})$.

Examples

Example 4

If $\mathbf{a} = \langle -2, 0, 3 \rangle$ and $\mathbf{b} = \langle 1, 4, 1 \rangle$, compute the scalar and vector projections of $\mathbf{a}$ and $\mathbf{b}$ on each other.

Solution. We have

$$\mathrm{comp}_{\mathbf{b}}(\mathbf{a}) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} = \frac{-2 + 0 + 3}{\sqrt{1^2 + 4^2 + 1^2}} = \frac{1}{3\sqrt{2}},$$

$$\mathrm{comp}_{\mathbf{a}}(\mathbf{b}) = \frac{\mathbf{b} \cdot \mathbf{a}}{|\mathbf{a}|} = \frac{-2 + 0 + 3}{\sqrt{(-2)^2 + 0^2 + 3^2}} = \frac{1}{\sqrt{13}},$$

$$\mathrm{proj}_{\mathbf{b}}(\mathbf{a}) = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2}\right)\mathbf{b} = \frac{1}{18}\langle 1, 4, 1 \rangle = \left\langle \frac{1}{18}, \frac{2}{9}, \frac{1}{18} \right\rangle,$$

$$\mathrm{proj}_{\mathbf{a}}(\mathbf{b}) = \left(\frac{\mathbf{b} \cdot \mathbf{a}}{|\mathbf{a}|^2}\right)\mathbf{a} = \frac{1}{13}\langle -2, 0, 3 \rangle = \left\langle \frac{-2}{13}, 0, \frac{3}{13}. \right\rangle$$

The *work* done by a force $\mathbf{F}$ acting through a displacement $\mathbf{d}$ is given by

$$W = \text{comp}_{\mathbf{d}}(\mathbf{F}) \cdot |\mathbf{d}| = \frac{\mathbf{F} \cdot \mathbf{d}}{|\mathbf{d}|}|\mathbf{d}| = \mathbf{F} \cdot \mathbf{d}.$$

Example 5

A sled is pulled 80 ft by a 30 lb force at an angle 40° above horizontal. How much work is done by this force?

Solution. Assuming the sled is pulled horizontally, we have:

$$W = \mathbf{F} \cdot \mathbf{d} = |\mathbf{F}| \cdot |\mathbf{d}| \cos 40^\circ = 30 \cdot 80 \cdot \cos 40^\circ \text{ ft lb}$$

$$= \boxed{2400 \cos 40^\circ \text{ ft lb} \approx 1838.5 \text{ ft lb}}$$