The Cross Product

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Calculus III

Introduction

- Today we will introduce our second vector-vector multiplication operation, namely the cross product.
- From an analytic point of view, the cross product is substantially more complicated than the dot product.
- Geometrically, however, it has a much more concrete interpretation.
- The definition of the cross product requires the use of determinants of (small) matrices, and this is where we will begin.

Determinants

A 2×2 *determinant* is given by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

A 3×3 *determinant* is given by

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

Examples

We have:

$$\begin{vmatrix} 4 & 5 \\ 2 & -1 \end{vmatrix} = 4 \cdot (-1) - 5 \cdot 2 = \; -14$$

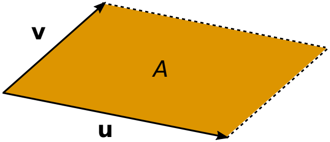
$$\begin{vmatrix} 3 & 7 \\ 2 & 5 \end{vmatrix} = 3 \cdot 5 - 7 \cdot 2 = 1$$

$$\begin{aligned} \begin{vmatrix} -2 & -5 & 1 \\ -4 & 3 & -1 \\ 3 & 3 & -2 \end{vmatrix} &= (-2) \begin{vmatrix} 3 & -1 \\ 3 & -2 \end{vmatrix} - (-5) \begin{vmatrix} -4 & -1 \\ 3 & -2 \end{vmatrix} + 1 \begin{vmatrix} -4 & 3 \\ 3 & 3 \end{vmatrix} \\ &= (-2)(-6 - (-3)) + 5(8 - (-3)) + (-12 - 9) \\ &= (-2)(-3) + 5 \cdot 11 - 21 = 40 \end{aligned}$$

What *Are* Determinants?

Determinants measure "geometric content."

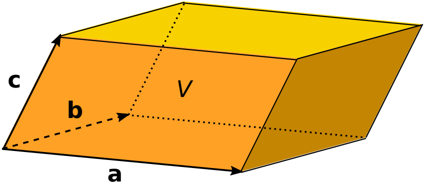
1. Two vectors $\mathbf{u} = \langle a, b \rangle$ and $\mathbf{v} = \langle c, d \rangle$ in $\mathbb{R}^2$ determine a parallelogram:



One can show its area A is given by

$$A = \left| \begin{vmatrix} a & b \\ c & d \end{vmatrix} \right| = |ad - bc|.$$

2. Three vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ and $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$ in $\mathbb{R}^3$ determine a *parallelepiped*:



Its volume is given by

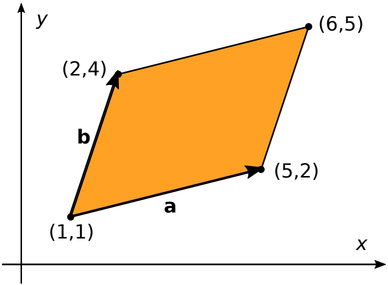
$$V = \left| \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \right|.$$

3. The sign of the determinant has to do with the relative orientation of the vectors involved.

Example 1

Find the area of the parallelogram with vertices $(1,1)$, $(2,4)$, $(6,5)$ and $(5,2)$.

Solution. Let's draw a sketch first:



We have

$$\mathbf{a} = \langle 5-1, 2-1\rangle = \langle 4, 1\rangle,$$

$$\mathbf{b} = \langle 2-1, 4-1\rangle = \langle 1, 3\rangle.$$

Therefore the area is (the absolute value of)

$$\begin{vmatrix} 4 & 1 \\ 1 & 3 \end{vmatrix} = 4 \cdot 3 - 1 \cdot 1 = \boxed{11.}$$

Remarks.

- If we had reversed the roles of **a** and **b**, the determinant would have been -11. The absolute value "corrects" the negative sign.
- One can pose analogous problems using the vertices of a parallelepiped in $\mathbb{R}^3$. In this case we would use a 3×3 determinant.

The Cross Product

Definition

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, their *cross product* is

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}.$$

Example. If $\mathbf{a} = \langle 1, 2, 3 \rangle$ and $\mathbf{b} = \langle -2, 0, 4 \rangle$, then

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -2 & 0 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 0 & 4 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 3 \\ -2 & 4 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ -2 & 0 \end{vmatrix} \mathbf{k}$$

$$= (8 - 0)\mathbf{i} - (4 + 6)\mathbf{j} + (0 + 4)\mathbf{k} = \boxed{\langle 8, -10, 4 \rangle.}$$

Properties of the Cross Product

- The cross product is only defined in $\mathbb{R}^3$. But we will sometimes apply it in two dimensions by treating $\mathbb{R}^2$ as the xy-plane in $\mathbb{R}^3$. That is, we treat $\langle a, b\rangle$ as $\langle a, b, 0\rangle$.
- $\mathbf{a} \times \mathbf{b}$ is a vector (as opposed to $\mathbf{a} \cdot \mathbf{b}$).
- The cross product "acts like" multiplication *in most ways*, e.g.

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c},$$
$$(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b}), \qquad \mathbf{a} \times \mathbf{0} = \mathbf{0}.$$

See section 12.4 for a more thorough list.

Properties (Cont.)

However, there are two notable differences between $\times$ and "ordinary" multiplication:

- $\times$ is *not* associative:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$$

  in general.

- $\times$ is *anti*-commutative:

$$\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}.$$

The second property follows from the fact that interchanging two rows in a determinant changes its sign. We will have a geometric interpretation shortly.

The Geometry of the Cross Product

1. Since

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \underbrace{\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}}_{x \text{ component}} \mathbf{i} - \underbrace{\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}}_{y \text{ component}} \mathbf{j} + \underbrace{\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}}_{z \text{ component}} \mathbf{k},$$

if $\mathbf{c} = \langle c_1, c_2, c_3\rangle$ and we replace $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ with c_1, c_2, c_3 (resp.), we immediately obtain

$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix},$$

so that

$$\boxed{|\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})| = \text{ volume of parallelepiped determined by } \mathbf{a}, \mathbf{b}, \mathbf{c}.}$$

2. Because of this, we must have

$$\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{a} \times \mathbf{b}) = 0,$$

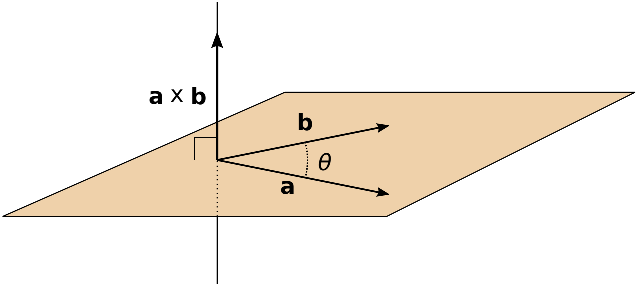
since the parallelepiped determined by only two vectors has no volume.

That is

$$\boxed{\mathbf{a} \times \mathbf{b} \text{ is orthogonal to both } \mathbf{a} \text{ and } \mathbf{b}.}$$

This *almost* determines the direction of $\mathbf{a} \times \mathbf{b}$ since there is only one line simultaneously perpendicular to two (nonparallel) vectors.

The precise direction of $\mathbf{a} \times \mathbf{b}$ is determined by the *right-hand rule:*

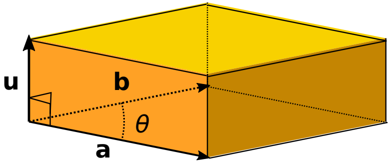


Note that if we interchange $\mathbf{a}$ and $\mathbf{b}$, the right-hand rule gives the opposite direction.

This (geometrically) explains why $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$.

3. Let $\mathbf{u}$ be a unit vector in the direction of $\mathbf{a} \times \mathbf{b}$.

Then $\mathbf{a}, \mathbf{b}, \mathbf{u}$ generate a right-angled parallelepiped:



According to elementary geometry

$$\begin{aligned} \text{Volume} &= (\text{height})(\text{area of base}) \\ &= \text{area of parallelogram determined by } \mathbf{a} \text{ and } \mathbf{b} \\ &= |\mathbf{a}| \cdot |\mathbf{b}| \cdot \sin\theta. \end{aligned}$$

But we also have

$$\text{Volume} = |\mathbf{u} \cdot (\mathbf{a} \times \mathbf{b})| = |\mathbf{u}| \cdot |\mathbf{a} \times \mathbf{b}| \cdot \cos 0 = |\mathbf{a} \times \mathbf{b}|.$$

Putting these together we conclude that

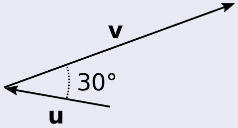
$$\boxed{|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| \cdot |\mathbf{b}| \cdot \sin\theta.}$$

We now have a complete geometric description of the cross product.

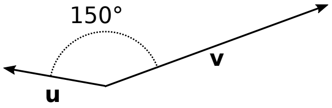
Examples

Example 2

Let $\mathbf{u}$ and $\mathbf{v}$ be the vectors shown below. If $|\mathbf{u}| = 3$ and $|\mathbf{v}| = 7$, describe $\mathbf{u} \times \mathbf{v}$.



Solution. We place $\mathbf{u}$ and $\mathbf{v}$ tail-to-tail to get the correct angle:



We find that $\mathbf{u} \times \mathbf{v}$ has magnitude

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}| \cdot |\mathbf{v}| \cdot \sin 150^\circ = 3 \cdot 7 \cdot \frac{1}{2} = \boxed{\frac{21}{2},}$$

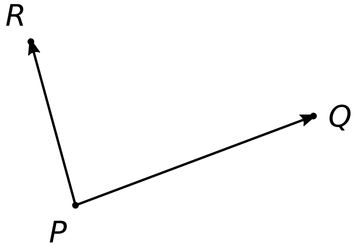
and the right-hand rule tells us that $\mathbf{u} \times \mathbf{v}$ points directly

$$\boxed{\text{into the page.}}$$

Example 3

Find a nonzero vector that is perpendicular to the plane containing the points $P = (2, 1, 5)$, $Q = (-1, 3, 4)$, $R = (3, 0, 6)$.

Solution. We take the plane containing P, Q and R as the page and make a rough sketch:



The vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ lie in the plane, so

$$\overrightarrow{PQ} \times \overrightarrow{PR}$$

will be perpendicular to it.

Since

$$\overrightarrow{PQ} = \langle -1-2, 3-1, 4-5\rangle = \langle -3, 2, -1\rangle,$$
$$\overrightarrow{PR} = \langle 3-2, 0-1, 6-5\rangle = \langle 1, -1, 1\rangle,$$

we find that

$$\begin{aligned}
\overrightarrow{PQ} \times \overrightarrow{PR} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 2 & -1 \\ 1 & -1 & 1 \end{vmatrix} \\
&= \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -3 & -1 \\ 1 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -3 & 2 \\ 1 & -1 \end{vmatrix}\mathbf{k} \\
&= \mathbf{i} - (-2)\mathbf{j} + \mathbf{k} \\
&= \boxed{\langle 1, 2, 1\rangle.}
\end{aligned}$$

Example 4

Is the point $S = (1,1,1)$ in the plane of the preceding example?

Solution. If S lies in the plane defined by P, Q, R, then the parallelepiped defined by the vectors $\overrightarrow{PQ}$, $\overrightarrow{PR}$, $\overrightarrow{PS}$ will be *flat.*

That is, its volume will be zero:

$$|\overrightarrow{PS} \cdot (\overrightarrow{PQ} \times \overrightarrow{PR})| = 0.$$

However, $\overrightarrow{PS} = \langle 1-2, 1-1, 1-5 \rangle = \langle -1, 0, -4 \rangle$, so that

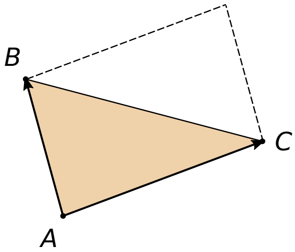
$$\overrightarrow{PS} \cdot (\overrightarrow{PQ} \times \overrightarrow{PR}) = \langle -1, 0, -4 \rangle \cdot \langle 1, 2, 1 \rangle = -5 \neq 0.$$

Thus $\boxed{S \text{ is not in the plane defined by } P, Q, R.}$

Example 5

Find the area of the triangle with vertices $A = (1, 2, 3)$, $B = (2, 0, -1)$, $C = (4, 1, -3)$.

Solution. The area of the triangle is half of the area of the parallelogram determined by $\overrightarrow{AB}$ and $\overrightarrow{AC}$:



This is given by $\frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$.

Since

$$\overrightarrow{AB} = \langle 2-1, 0-2, -1-3\rangle = \langle 1, -2, -4\rangle,$$
$$\overrightarrow{AC} = \langle 4-1, 1-2, -3-3\rangle = \langle 3, -1, -6\rangle,$$

we find that

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & -4 \\ 3 & -1 & -6 \end{vmatrix} = \begin{vmatrix} -2 & -4 \\ -1 & -6 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -4 \\ 3 & -6 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix} \mathbf{k}$$
$$= 8\mathbf{i} - 6\mathbf{j} + 5\mathbf{k} = \langle 8, -6, 5\rangle.$$

Thus

$$\frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\sqrt{8^2 + (-6)^2 + 5^2} = \frac{1}{2}\sqrt{125} = \boxed{\frac{5\sqrt{5}}{2}.}$$