# Lines and Planes in $\mathbb{R}^3$

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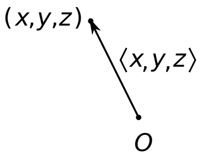
Calculus III

# Introduction

Our goal today is to find equations representing lines and planes in space ($\mathbb{R}^3$).

The geometry and arithmetic of vectors will be our main tools.

We define the *position vector* of a point $(x, y, z)$ to be the vector $\langle x, y, z\rangle$, the vector that "points to" $(x, y, z)$:



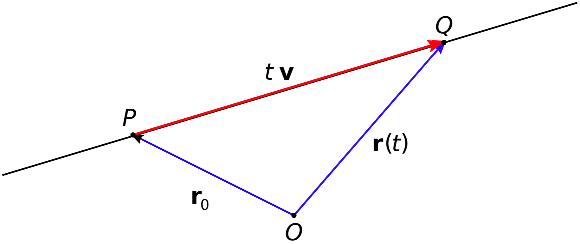
# Lines in $\mathbb{R}^3$

A line in $\mathbb{R}^3$ is determined by two pieces of data:

- A point $P = (x_0, y_0, z_0)$ on the line;
- A direction vector $\mathbf{v} = \langle a, b, c \rangle$.

Let $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$ be the position vector of $P$.

Let $Q = (x, y, z)$ be any other point on the line, and introduce the origin $O$.


$Q$
$t\,\mathbf{v}$
$P$
$\mathbf{r}(t)$
$\mathbf{r}_0$
$O$

From the diagram we see that the position vector of $Q$ is given by the *vector equation*

$$\boxed{\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}, \quad t \in \mathbb{R}.}$$

In terms of components we have

$$\begin{aligned}\mathbf{r}(t) &= \mathbf{r}_0 + t\mathbf{v} \\ &= \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle \\ &= \langle x_0 + at, y_0 + bt, z_0 + ct \rangle,\end{aligned}$$

which tells us that $Q$ can also be given by the *parametric equations*

$$\boxed{\begin{aligned} x &= x_0 + at, \\ y &= y_0 + bt, \\ z &= z_0 + ct. \end{aligned}}$$

# Examples

## Example 1

Find a vector equation for the line through $P = (-1, 2, 3)$ and $Q = (2, -2, 5)$.

*Solution.* We need two things: a point on the line and the direction (vector) of the line.

We take $P = (-1, 2, 3)$ as our base point.

The vector $\overrightarrow{PQ} = \langle 2 - (-1), -2 - 2, 5 - 3 \rangle = \langle 3, -4, 2 \rangle$ gives the correct direction.

So the line is given by

$$\mathbf{r}(t) = \langle -1, 2, 3 \rangle + t\langle 3, -4, 2 \rangle = \langle -1 + 3t, 2 - 4t, 3 + 2t \rangle.$$

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## Example 2

Find a vector equation for the line through $(5, -6, 7)$ that is parallel to the line with parametric equations $x = 1 + t$, $y = 2$, $z = 3 + 2t$.

*Solution.* We need two things: a point on the line and the direction (vector) of the line.

We are given $P = (5, -6, 7)$ on the line.

To find the direction vector of the given line we simply read the coefficients of $t$ in each given component:

$$\mathbf{v} = \langle 1, 0, 2 \rangle.$$

So the line is given by

$$\boxed{\mathbf{r}(t) = \langle 5, -6, 7 \rangle + t\langle 1, 0, 2 \rangle = \langle 5 + t, -6, 7 + 2t \rangle.}$$

## Example 3

Find the point of intersection of the lines from the preceding examples.

*Solution.* The two lines are given by

$$\mathbf{r}_1(t) = \langle -1 + 3t, 2 - 4t, 3 + 2t \rangle,$$
$$\mathbf{r}_2(t) = \langle 5 + t, -6, 7 + 2t \rangle.$$

They *intersect* when $\mathbf{r}_1(t) = \mathbf{r}_2(s)$ for some $r, s \in \mathbb{R}$. Equating components gives the system

$$-1 + 3t = 5 + s,$$
$$2 - 4t = -6,$$
$$3 + 2t = 7 + 2s.$$

The middle equation gives $t = 2$. But we must be sure *both* of the other equations can be solved simultaneously.

With $t = 2$ the first becomes $5 = 5 + s$, so that $s = 0$.

And with $t = 2$, $s = 0$, the third is $7 = 7$, which is valid.

So the lines intersect at (the point with position vector)

$$\mathbf{r}_1(2) = \mathbf{r}_2(0) = \boxed{\langle 5, -6, 7 \rangle}.$$

## Example 4

Show that the lines

$$\mathbf{r}_1(t) = \langle 1 + t, -3 - t, 5 + 2t \rangle,$$

$$\mathbf{r}_2(s) = \langle 4 - s, -3 + s, 6 + 2s \rangle,$$

are *skew* (neither intersecting nor parallel).

*Solution.* By reading the coefficients of $s$ and $t$ we find that the directions of the two lines are

$$\mathbf{u} = \langle 1, -1, 2 \rangle,$$
$$\mathbf{v} = \langle -1, 1, 2 \rangle.$$

Since $\mathbf{u}$ is not a scalar multiple of $\mathbf{v}$ (why?), the two lines are not parallel.

It remains to show that they do not intersect. We set $\mathbf{r}_1(t) = \mathbf{r}_2(s)$ and show this leads to a contradiction.

Equating components yields the system

$$\begin{aligned} 1 + t &= 4 - s, \\ -3 - t &= -3 + s, \\ 5 + 2t &= 6 + 2s. \end{aligned}$$

The first two equations can be rearranged to

$$\begin{aligned} s + t &= 3, \\ s + t &= 0, \end{aligned}$$

which is clearly impossible. So the lines do not intersect.

# Planes in $\mathbb{R}^3$

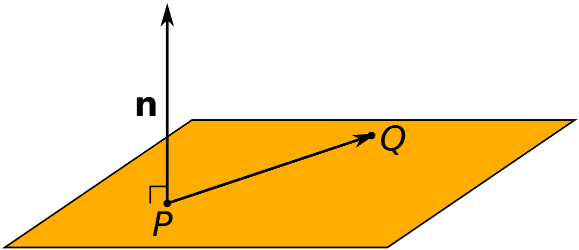
A plane in $\mathbb{R}^3$ is determined by two pieces of data:

- A point $P = (x_0, y_0, z_0)$ on the plane;
- A *normal vector* $\mathbf{n} = \langle a, b, c \rangle$.

The normal vector specifies which way the plane "faces."

Let $Q = (x, y, z)$ be any point on the plane.

The vector $\overrightarrow{PQ} = \langle x - x_0, y - y_0, z - z_0 \rangle$ lies in the plane, and is therefore orthogonal to $\mathbf{n}$.


n
Q
P

So we must have $\overrightarrow{PQ} \cdot \mathbf{n} = 0$ or

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.}$$

**Remarks.**

1. By distributing $a, b, c$ through the parentheses, we can always put any plane equation into the form

$$ax + by + cz + d = 0.$$

2. Conversely, any equation of the form $ax + by + cz + d = 0$ represents a plane with normal vector

$$\mathbf{n} = \langle a, b, c \rangle.$$

# Examples

## Example 5

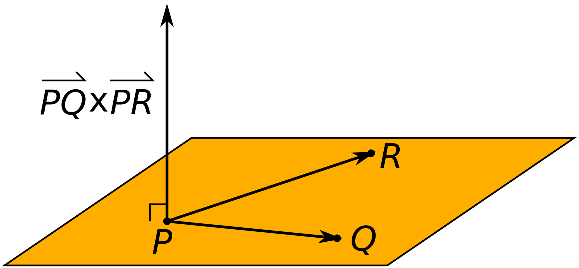
Find an equation for the plane containing $P = (1, 2, 3)$, $Q = (-2, 4, 1)$ and $R = (0, 6, -2)$.

*Solution.* We need two things: a point on the plane and the normal vector.

We have no shortage of points. $P = (1, 2, 3)$ will work.

The vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ both lie in the plane.

Therefore $\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$ will serve as the normal vector.


$\overrightarrow{PQ} \times \overrightarrow{PR}$
$R$
$P$
$Q$

We have

$$\overrightarrow{PQ} = \langle -2-1, 4-2, 1-3 \rangle = \langle -3, 2, -2 \rangle,$$
$$\overrightarrow{PR} = \langle 0-1, 6-2, -2-3 \rangle = \langle -1, 4, -5 \rangle,$$

so that

$$\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 2 & -2 \\ -1 & 4 & -5 \end{vmatrix}$$
$$= -2\mathbf{i} - 13\mathbf{j} - 10\mathbf{k} = -\langle 2, 13, 10 \rangle.$$

We can drop the negative sign (why?), so that the plane equation is

$$2(x-1) + 13(y-2) + 10(z-3) = 0,$$

or

$$\boxed{2x + 13y + 10z - 58 = 0.}$$

## Example 6

Show that the planes $2x - 5y + 9z = 6$ and $4x - 10y + 11z = 0$ are not parallel. Find parametric equations for their line of intersection.

*Solution.* Two planes are parallel iff their normal vectors are parallel.

By reading the coefficients, we find that the normals here are

$$\begin{aligned} \mathbf{n_1} &= \langle 2, -5, 9 \rangle, \\ \mathbf{n_2} &= \langle 4, -10, 11 \rangle. \end{aligned}$$

As these are not scalar multiples of one another (why not?), they are not parallel.

It follows that the two planes have a common line of intersection.

Since its direction is *in* both planes, it must be *orthogonal* to *both* normals.

The direction of the line is therefore

$$\mathbf{n_1} \times \mathbf{n_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -5 & 9 \\ 4 & -10 & 11 \end{vmatrix} = \langle 35, 14, 0 \rangle = 7 \underbrace{\langle 5, 2, 0 \rangle}_{\mathbf{v}}.$$

Note that this means the line will be parallel to the $xy$-plane.

We still need a point on the line of intersection.

This requires us to solve both plane equations simultaneously.

Since they are equations in 3 variables, we'll need to specify one of them, say $x = 0$.

The plane equations then become

$$\begin{aligned} -5y + 9z &= 6, \\ -10y + 11z &= 0. \end{aligned}$$

Variable elimination yields $y = \frac{66}{35}$ and $z = \frac{12}{7}$. So we get the point

$$\left(0, \frac{66}{35}, \frac{12}{7}\right).$$

Our line is therefore given by

$$\begin{aligned}\mathbf{r}(t) &= \mathbf{r}_0 + t\mathbf{v}\\ &= \left\langle 0, \frac{66}{35}, \frac{12}{7} \right\rangle + t\langle 5, 2, 0\rangle\\ &= \left\langle 5t, \frac{66}{35} + 2t, \frac{12}{7} \right\rangle,\end{aligned}$$

or in parametric form

$$\boxed{\begin{aligned} x &= 5t,\\ y &= \frac{66}{35} + 2t,\\ z &= \frac{12}{7}. \end{aligned}}$$

## Example 7

Show that the planes $3x - 2y + z = 12$ and $x + 3y - 5z = 7$ are not parallel, and find the acute angle between them.

*Solution.* The normal vectors are:

$$\mathbf{n}_1 = \langle 3, -2, 1 \rangle,$$
$$\mathbf{n}_2 = \langle 1, 3, -5 \rangle.$$

Since these are not scalar multiples of one another (why not?), the planes are not parallel.

The angle between the planes will be the same as the angle between the normal vectors.

We can find the latter with the dot product.

We have

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| \cdot |\mathbf{n}_2|} = \frac{3 \cdot 1 + (-2) \cdot 3 + 1 \cdot (-5)}{\sqrt{3^2 + (-2)^2 + 1^2}\sqrt{1^2 + 3^2 + (-5)^2}}$$

$$= \frac{-8}{7\sqrt{10}} < 0.$$

Because this is negative, the angle $\theta$ is obtuse, so we actually need its supplement:

$$\phi = \pi - \theta = \boxed{\pi - \arccos\left(\frac{-8}{7\sqrt{10}}\right) \approx 68.8^\circ.}$$