Cylinders and Quadric Surfaces

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Calculus III

Introduction

We have seen that every linear equation $ax + by + cz = d$ in (x, y, z) represents a plane in $\mathbb{R}^3$.

A plane is our first example of a *surface*: a set which is "locally planar."

Every "nice" equation in (x, y, z), such as

$$(x - 3)^2 + (y + 1)^2 + (z - 1)^2 = 25,$$

represents a surface in $\mathbb{R}^3$.

Today we will study two special classes of surfaces: the *cylinders* and *quadric surfaces*.

Cylinders

Definition

A *cylinder* is any surface created by "dragging" a 2D curve along an axis (in $\mathbb{R}^3$).

Examples.

- The surface $y = x^2$ is a cylinder in $\mathbb{R}^3$ along the z-axis since it is "missing" the variable z.
- The surface $y^2 + 2z^2 = 1$ is a cylinder in $\mathbb{R}^3$ along the x-axis since it is "missing" the variable x.
- The surface $z = \sin(x + y)$ is also a cylinder, although this is harder to identify (see Maple diagram).

Quadric Surfaces

One degree up from the "linear surfaces" (planes) we have:

Definition

A *quadric surface* is given by an equation of the form

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0,$$

where $A, B, C, \dots$ are real numbers for which at least one of $A, B, \dots, F$ is nonzero.

Via translations and rotations, one can always reduce the equation of a quadric surface to

$$Ax^2 + By^2 + Cz^2 + J = 0, \tag{1}$$

or

$$Ax^2 + By^2 + Iz = 0. \tag{2}$$

Goal: Describe the graphs of the "generic" quadric surfaces (1) and (2).

Up to signs, there are 6 cases to consider:

- $\underline{J = 0}$: Up to permutation of the variables, these surfaces have the form

$$\boxed{\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}.}$$

- $\underline{J \neq 0}$: We can rearrange (1) to one of the following forms:
  * $\boxed{\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1;}$
  * $\boxed{\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1;}$
  * $\boxed{-\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1;}$
- $\underline{I \neq 0}$: Writing $I = 1/c$, there are two possible situations:
  * $\boxed{\dfrac{z}{c} = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2};}$
  * $\boxed{\dfrac{z}{c} = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}.}$

Graphs of Quadric Equations

Example 1

Describe the quadric surface given by the equation $z^2 = \frac{x^2}{9} + \frac{y^2}{4}$.

Solution. We take cross sections (*traces*) parallel to the xy-plane.

This amounts to setting $z = k$, a constant:

$$k^2 = \frac{x^2}{9} + \frac{y^2}{4} \quad \Leftrightarrow \quad \frac{x^2}{(3k)^2} + \frac{y^2}{(2k)^2} = 1.$$

This is an ellipse with axes $3|k|$ and $2|k|$.

When $k = 0$ we just get a point.

As we increase $|k|$, the dimensions $2|k|$ and $3|k|$ grow linearly.

Stacking these together we obtain a *cone* (see Maple diagram).

Example 2

Describe the quadric surface given by $x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$.

Solution. Again we take cross sections parallel to the xy plane by setting $z = k$ (constant):

$$x^2 + \frac{y^2}{4} + \frac{k^2}{9} = 1 \quad \Leftrightarrow \quad \frac{x^2}{1 - k^2/9} + \frac{y^2}{4(1 - k^2/9)} = 1.$$

We again obtain ellipses, but this time with axes $\sqrt{1 - k^2/9}$ and $2\sqrt{1 - k^2/9}$.

Note that these *decrease* as we increase $|k|$, equal zero when $|k| = 3$, and are *imaginary* for $|k| > 3$.

So we only have cross sections for $|k| \leq 3$. Stacking them together yields an *ellipsoid* (see Maple diagram).

Example 3

Describe the quadric surface given by $\dfrac{x^2}{4} + y^2 - \dfrac{z^2}{16} = 1$.

Solution. Once again we take cross sections by setting $z = k$.

This time we obtain

$$\frac{x^2}{4} + y^2 - \frac{k^2}{16} = 1 \iff \frac{x^2}{4(1 + k^2/16)} + \frac{y^2}{1 + k^2/16} = 1.$$

Again we have a family of ellipses, this time with dimensions $2\sqrt{1 + k^2/16}$ and $\sqrt{1 + k^2/16}$.

They are as small as possible (thought not just a point) when $k = 0$, and grow roughly linearly with $|k|$.

Stacking them together we get a *one-sheeted hyperboloid* (see Maple diagram).

Example 4

Describe the quadric surface given by $-x^2 - \frac{y^2}{9} + \frac{z^2}{3} = 1$.

Solution. This time when we take cross sections by setting $z = k$ we find that

$$-x^2 - \frac{y^2}{9} + \frac{k^2}{3} = 1 \iff \frac{x^2}{k^2/3 - 1} + \frac{y^2}{9(k^2/3 - 1)} = 1.$$

We again have ellipses, this time with axes $\sqrt{k^2/3 - 1}$ and $3\sqrt{k^2/3 - 1}$.

These grow roughly linearly in $|k|$, but are *imaginary* if $|k| < \sqrt{3}$. So the surface only exists for $|z| \geq \sqrt{3}$, and has two pieces.

Stacking the cross sections we end up with a *two-sheeted hyperboloid* (see Maple diagram).

Example 5

Describe the quadric surface given by the equation $z = 5x^2 + 7y^2$.

Solution. Setting $z = k$ now gives us

$$5x^2 + 7y^2 = k \iff \frac{5x^2}{k} + \frac{7y^2}{k} = 1$$

which only has solutions provided $k \geq 0$.

So there are *no* cross sections below the xy plane, and those above are ellipses with dimensions $\sqrt{k/5}$ and $\sqrt{k/7}$.

These grow parabolically as k increases, so stacking them gives us a *paraboloid* (see Maple diagram).

Example 6

Describe the quadric surface given by the equation $z = x^2 - y^2$.

Solution. Setting $z = k$ we obtain

$$x^2 - y^2 = k \;\Leftrightarrow\; \frac{x^2}{k} - \frac{y^2}{k} = 1.$$

With one exception, these are *hyperbolas*: opening left/right with x-intercepts $\pm\sqrt{k}$ when $k > 0$, and opening up/down with y-intercepts $\pm\sqrt{-k}$ when $k < 0$.

When $k = 0$ we get

$$x^2 - y^2 = 0 \;\Leftrightarrow\; (x - y)(x + y) = 0,$$

which is the pair of lines $x = y$ and $x = -y$.

Stacking these together we end up with a *hyperbolic paraboloid* (see Maple diagram).