Functions of Several Variables

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Calculus III

Introduction

Today we will begin studying functions of several variables.

We will focus on elementary concepts such as domains and graphs.

We will see that visualizing functions of two or more variables can be challenging:

- The graph of a two-variable function is a *surface* in $\mathbb{R}^3$.

- The graph of a function of three or more variables cannot easily be embedded into 3-dimensional space.

A function f of n variables has n real inputs (a point in $\mathbb{R}^n$) and a single (real) output.

We write $f(x_1, x_2, \ldots, x_n)$ for the value of f at the point $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

If f is given by a particular formula, we take its *domain* to be the largest subset of $\mathbb{R}^n$ on which that formula is defined.

Remark. For functions of a single variable (Calc. I), domains are frequently intervals. In two or more variables they can be more complicated regions.
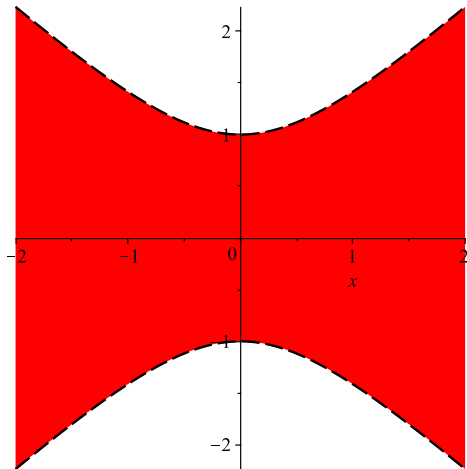
Examples

Example 1

Find and sketch the domain of $f(x, y) = \ln(1 - y^2 + x^2)$.

Solution. Recall that $\ln t$ is defined iff $t > 0$. So the domain of f is given by

$$1 - y^2 + x^2 > 0 \quad \Leftrightarrow \quad y^2 - x^2 < 1. \tag{1}$$

The equation $y^2 - x^2 = 1$ represents a hyperbola.

Since $(0, 0)$ satisfies (1), the domain must be the region *between* the two branches of the hyperbola.
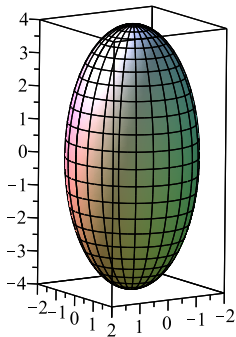
The region $y^2 - x^2 < 1$.

Example 2

Find and sketch the domain of $g(x, y, z) = \sqrt{16 - 4x^2 - 4y^2 - z^2}$.

Solution. Since $\sqrt{t}$ is defined only for $t \geq 0$, the domain is given by

$$16 - 4x^2 - 4y^2 - z^2 \geq 0 \iff 4x^2 + 4y^2 + z^2 \leq 16$$

$$\iff \frac{x^2}{4} + \frac{y^2}{4} + \frac{z^2}{16} \leq 1.$$

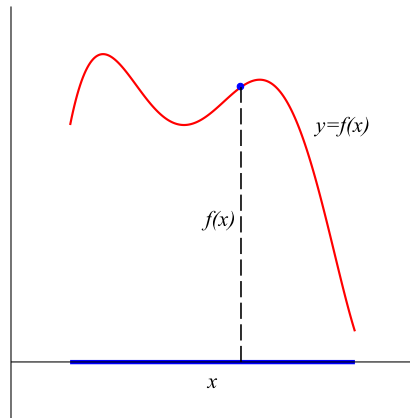The points satisfying $\frac{x^2}{4} + \frac{y^2}{4} + \frac{z^2}{16} = 1$ form an ellipsoid.

Since $(0, 0, 0)$ satisfies $\frac{x^2}{4} + \frac{y^2}{4} + \frac{z^2}{16} \leq 1$, the domain must be the (solid) region inside and on the ellipsoid.

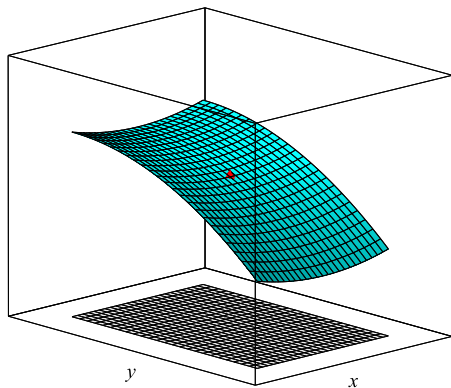The ellipsoid $\frac{x^2}{4} + \frac{y^2}{4} + \frac{z^2}{16} = 1$.

Visualizing Functions of Several (2 or 3) Variables

Recall: The graph of a function $f(x)$ of one variable is the curve in the xy-plane given by $y = f(x)$:

The blue line segment represents the domain of f and the indicated point has coordinates $(x, f(x))$.

Proceeding by analogy, given a function $f(x, y)$, its *graph* is the surface in $\mathbb{R}^3$ with equation $z = f(x, y)$:

The gray rectangle is the domain of f (in the xy-plane) and the red point has coordinates $(x, y, f(x, y))$.

Level Sets

As with quadric surfaces, we can use cross sections to help us understand the graph of a function of several variables.

Definition

Given $f(x_1, x_2, \ldots, x_n)$, a *level set* of f is the collection of points in $\mathbb{R}^n$ satisfying $f(x_1, x_2, \ldots, x_n) = k$, where k is a constant.

In two variables a level set is a *curve* of the form

$$f(x, y) = k,$$

called a *contour* of f.

In three variables a level set is a *surface* of the form

$$f(x, y, z) = k,$$

called a *level surface* of f.

Examples

Example 3

Describe the contours of the function $f(x, y) = y - x^2$.

Solution. We set $f(x, y) = y - x^2 = k$. Rearranging gives

$$y = x^2 + k,$$

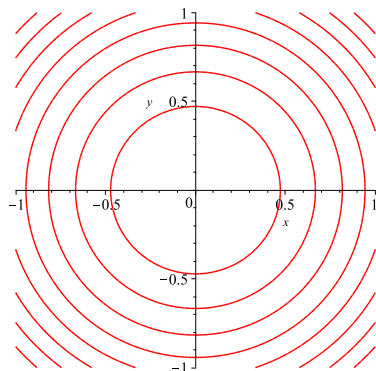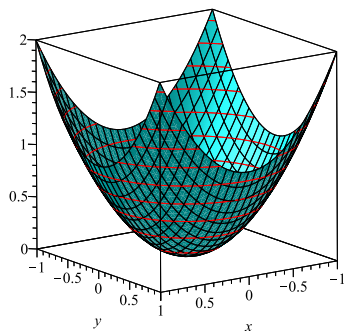which are *upward opening parabolas*.

Example 4

Describe the level surfaces of the function
$f(x, y, z) = x^2 + y^2 + z^2$.

Solution. We set $f(x, y, z) = x^2 + y^2 + z^2 = k$. For $k > 0$ these are *spheres*.

Contour Maps

A *contour map* of a function $f(x, y)$ is a "topographical map" of the graph $z = f(x, y)$ which includes multiple contours of f (with evenly spaced values of k).

Examples

In the following examples:

- Determine and sketch the domain of f.
- Describe the contours of f.
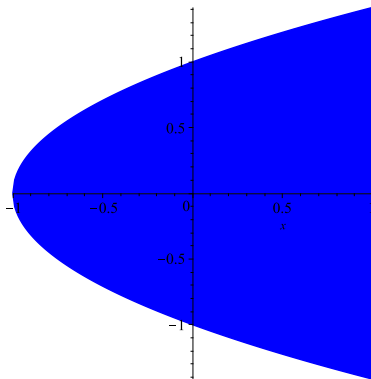- Create a contour map of f and compare it to the graph of f.

Example 5

$f(x, y) = \sqrt{1 + x - y^2}$

Solution. The domain is given by

$$\boxed{1 + x - y^2 \geq 0} \tag{2}$$

The curve $1 + x - y^2 = 0$ is a parabola, and $(0, 0)$ satisfies (2).

So the domain is the region to the *right* of the parabola:

The contours are given by

$$\sqrt{1 + x - y^2} = k \iff 1 + x - y^2 = k^2 \iff x = y^2 + k^2 - 1,$$

which are rightward opening parabolas with x-intercepts $x = k^2 - 1$. As $k \geq 0$ increases, these move to the right at an ever increasing rate. See Maple diagram.
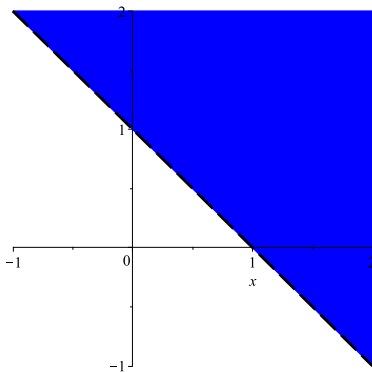
Example 6

$f(x, y) = \ln(x + y - 1)$.

Solution. The domain is given by

$$\boxed{x + y - 1 > 0.} \tag{3}$$

The equation $x + y - 1 = 0$ represents a line, and $(0, 0)$ *fails* (3).

So the domain is the region *below* the line $x + y - 1 = 0$:

The contours have the form

$$\ln(x + y - 1) = k \iff x + y - 1 = e^k \iff y = 1 + e^k - x,$$

which are lines of slope -1 with y-intercepts $y = 1 + e^k$.

As $k \to \infty$, the contours move up the y-axis rapidly.

As $k \to -\infty$, the contours "bunch up" and approach the line $y = 1 - x$.

This tells us that the graph of f flattens out as we move away from the origin, and becomes extremely steep as we approach the line $y = 1 - x$. See Maple diagram.

Example 7

$f(x, y) = 9x^2 + 4y^2$

Solution. The domain is all of $\boxed{\mathbb{R}^2.}$

The contours have the form

$$9x^2 + 4y^2 = k \iff \frac{9x^2}{k} + \frac{4y^2}{k} = 1,$$

which are *ellipses* with dimensions $\sqrt{k}/3$ and $\sqrt{k}/2$ (note we must have $k \geq 0$).

These grow away from the origin at an ever *decreasing* rate, and so tend to "bunch up" as $k \to \infty$.

This tells us that the graph (which we know to be a paraboloid) gets ever *steeper* as we move outward. See Maple diagram.
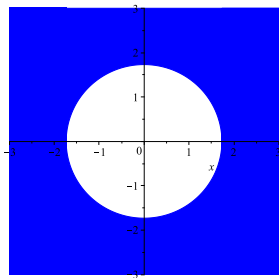
Example 8

$$f(x,y) = \sqrt{x^2 + y^2 - 3}$$

Solution. The domain is given by

$$x^2 + y^2 - 3 \geq 0 \;\Leftrightarrow\; \boxed{x^2 + y^2 \geq 3,}$$

which is everything *outside* the circle of radius $\sqrt{3}$ centered at $(0,0)$:

The contours have the form

$$\sqrt{x^2 + y^2 - 3} = k \;\Leftrightarrow\; x^2 + y^2 - 3 = k^2 \;\Leftrightarrow\; x^2 + y^2 = k^2 + 3,$$

which are *circles* of radius $\sqrt{k^2 + 3}$.

As $k \geq 0$ increases, the radius starts at $\sqrt{3}$ and increases roughly linearly when k is large.

This means the contours become evenly spaced as we move away from the origin. See Maple diagram.

Example 9

$$f(x, y) = \frac{y}{x^2 + y^2}$$

Solution. The denominator $x^2 + y^2$ is zero only at $(0, 0)$. It follows that the domain is the *punctured plane*

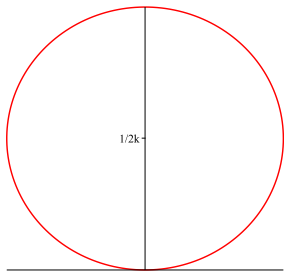$$\boxed{\mathbb{R}^2 \setminus \{(0, 0)\}.}$$

For $k \neq 0$ the contours have the form

$$\frac{y}{x^2 + y^2} = k \;\; \Leftrightarrow \;\; \frac{y}{k} = x^2 + y^2 \;\; \Leftrightarrow \;\; x^2 + y^2 - \frac{y}{k} = 0.$$

Completing the square in y yields

$$x^2 + \left(y - \frac{1}{2k}\right)^2 - \frac{1}{4k^2} = 0 \;\; \Leftrightarrow \;\; x^2 + \left(y - \frac{1}{2k}\right)^2 = \frac{1}{4k^2}.$$

These are *circles* with center $\left(0, \dfrac{1}{2k}\right)$ and radius $\dfrac{1}{2|k|}$:

For $k < 0$ the circles are *below* the y-axis.

As $|k| \to 0$ the radius and center become extremely large, indicating that the graph approaches the xy-plane as we move away from the origin.

As $|k| \to \infty$ the radius and center shrink to 0, indicating that the graph has two "spikes" at the origin.

See Maple diagram.