Limits in $\mathbb{R}^2$ and $\mathbb{R}^3$

Ryan C. Daileda

Trinity University

Calculus III

Introduction

Today marks the beginning of our study of the calculus of functions of several variables.

We will start where most treatments of Calculus I begin: with limits and continuity.

We will see that limits in multiple variables can pose significant technical (and psychological) challenges.

Once we've considered limits thoroughly, we will use them to define continuity in several variables.

Recall. For a function $f(x)$ of a single variable,

$$\lim_{x \to a} f(x) = L \tag{1}$$

means that as x approaches (but *does not equal*) a, the values $f(x)$ approach L.

Somewhat more precisely, we can make $f(x)$ as close to L as we choose by making x appropriately close to (but *not equal to*) a.

Here "close to" refers to distance on the real axis.

Because we have a measure of distance in $\mathbb{R}^2$ (and $\mathbb{R}^3$), the notion of "close to" makes sense there, too.

This gives us a way of talking about limits of functions of two (or more) variables.

Definition

We will write

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

to mean that as (x, y) approaches (but *does not equal*) (a, b), the values $f(x, y)$ approach L.

Remarks.

- We will use $\to$ as shorthand for "approaches."
- In terms of distance, $(x, y) \to (a, b)$ provided

$$\sqrt{(x-a)^2 + (y-b)^2} \to 0.$$

- We can take limits in $\mathbb{R}^n$ for any n by using the appropriate distance formula.

Examples

Example 1

Evaluate $\lim_{(x,y)\to(0,0)} \dfrac{x^4 - y^4}{x^2 - y^2}$.

Remark. Notice that $(x, y) \to (0, 0)$ means $\sqrt{x^2 + y^2} \to 0$.

Solution. We have

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{x^4 - y^4}{x^2 - y^2} &= \lim_{(x,y)\to(0,0)} \frac{(x^2 - y^2)(x^2 + y^2)}{x^2 - y^2} \\
&= \lim_{(x,y)\to(0,0)} x^2 + y^2 \\
&= \lim_{(x,y)\to(0,0)} \left(\sqrt{x^2 + y^2}\right)^2 = 0^2 = \boxed{0.}
\end{aligned}$$

In the preceding example we algebraically cancelled the “zero” in the denominator in order to evaluate the limit, much as in Calc. I.

This isn’t always possible in general, and we have to rely on “approximate algebra” (inequalities) to get the job done.

Example 2

Evaluate $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2}$.

Solution. Let’s look at the the function $\dfrac{x^2 y}{x^2+y^2}$. It is not defined at $(0,0)$ because x^2+y^2 vanishes there.

However, no amount of algebra will introduce a factor of $x^2 + y^2$ in the numerator (to cancel the denominator).

Instead, we proceed as follows. We have

$$0 \leq \left| \frac{x^2 y}{x^2 + y^2} \right| = \frac{x^2 |y|}{x^2 + y^2} \leq \frac{(x^2 + y^2)|y|}{x^2 + y^2} = |y|.$$

We now have $\dfrac{x^2 y}{x^2 + y^2}$ “squeezed” between 0 and $|y|$.

Since $|y| \to 0$ as $(x, y) \to (0, 0)$, we must have

$$\boxed{\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2 + y^2} = 0,}$$

by the *Squeeze Theorem*. $\square$

Example 3

Evaluate $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$.

Solution. We have

$$0 \le \left|\frac{xy}{\sqrt{x^2+y^2}}\right| = \frac{|x|\cdot|y|}{\sqrt{x^2+y^2}} = \frac{|x|\sqrt{y^2}}{\sqrt{x^2+y^2}} \le \frac{|x|\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} = |x|.$$

That is, $\dfrac{xy}{\sqrt{x^2+y^2}}$ is "squeezed" between 0 and $|x|$.

Since $|x| \to 0$ as $(x,y) \to (0,0)$, we conclude that

$$\boxed{\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0.}$$

Remark. In the majority of out limits we will have $(x, y) \to (0, 0)$, simply out of convenience.

Example 4

Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y}{x^2 + y^2}$ does not exist (DNE).

Remark. We have seen that $\dfrac{y}{x^2 + y^2}$ behaves "strangely" at the origin, so this shouldn't be a surprise.

Solution. We approach $(0, 0)$ in two different ways and get different results.

We will let $(x, y) \to (0, 0)$ *along the coordinate axes.*

Along the positive x-axis we have $y = 0$ and $x \to 0^+$. Thus

$$\frac{y}{x^2 + y^2} = \frac{0}{0^2 + x^2} = 0 \to 0.$$

(*Note:* This is *not* a "$\frac{0}{0}$" limit because $x > 0$.)

Along the positive y-axis we have $x = 0$ and $y \to 0^+$. Thus

$$\frac{y}{x^2 + y^2} = \frac{y}{y^2 + 0^2} = \frac{1}{y} \to \infty.$$

Because $0 \neq \infty$, the limit $\boxed{\text{DNE.}}$

Example 5

Show that $\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+2y^2}$ does not exist.

Solution. Again, we approach $(0,0)$ from two different directions and get different results.

This time, though, along either the x-axis ($y=0$) or the y-axis ($x=0$) we have

$$\frac{2xy}{x^2+2y^2} = 0 \to 0$$

as $(x,y)\to(0,0)$ (the numerator is 0 and the denominator is not).

So we need to check another line through $(0,0)$. Along $y = x$ we have

$$\frac{2xy}{x^2+2y^2} = \frac{2x^2}{x^2+2x^2} = \frac{2}{3} \to \frac{2}{3}$$

as $(x,y) \to (0,0)$.

Since $0 \neq \frac{2}{3}$, the limit $\boxed{\text{DNE.}}$ $\square$

These examples suggest a connection between general limits and "directional" limits.

Recall. If $f(x)$ is a function of one variable, then

$$\lim_{x \to a} f(x) = L \iff \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L.$$

That is, the "two-sided" limit exists iff the two one-sided limits agree.

There is a similar connection in higher dimensions (more variables), but the situation is somewhat more complicated.

This is because in one dimension ($\mathbb{R}$) there are only two ways to approach a point: from the left, or from the right.

But in $\mathbb{R}^2$ (or $\mathbb{R}^3$) we can approach a point from an *infinitude* of directions (any vector in $\mathbb{R}^2$).

Moreover, we can approach a point in ways that *aren't straight lines*, e.g. we might approach along a parabola, or along a spiral, or...

To test whether or not a limit exists "directionally" in $\mathbb{R}^2$ (or $\mathbb{R}^3$) we must test *every possible* path of approach.

Theorem 1

We have $\lim_{(x,y)\to(a,b)} f(x,y) = L$ *iff* $f(x,y) \to L$ *along* every *curve through* (a,b).

Remarks.

- This is almost never used to show limits exist, because it is almost impossible to check *every* curve through a point.
- Instead we usually apply the converse: $\lim_{(x,y)\to(a,b)} f(x,y)$ DNE iff $f(x,y)$ approaches two *different* values along two *different* curves through (a,b).

The following example demonstrates why "checking straight lines" is *not sufficient* to show that a limit in $\mathbb{R}^2$ exists.

Example 6

Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^6+y^2}$ DNE.

Solution. Let's be efficient. Rather than check lines through $(0,0)$ randomly, let's check them "all at once."

A (non-vertical) line through the origin has equation $y = mx$ for some m.

If we approach $(0,0)$ along $y = mx$ we have

$$\frac{x^3y}{x^6+y^2} = \frac{x^3(mx)}{x^6+(mx)^2} = \frac{mx^4}{x^6+m^2x^2} = \frac{mx^2}{x^4+m^2} \to \frac{0}{0^4+m^2} = 0.$$

That is:

$$\frac{x^3 y}{x^6 + y^2} \to 0 \text{ along every straight line through } (0,0)$$

(strictly speaking, the y-axis still needs to be checked).

However, suppose we approach $(0,0)$ along the cubic curve $y = x^3$:

$$\frac{x^3 y}{x^6 + y^2} = \frac{x^3 \cdot x^3}{x^6 + (x^3)^2} = \frac{x^6}{2x^6} = \frac{1}{2} \to \frac{1}{2}.$$

Since $0 \neq \frac{1}{2}$, the limit $\boxed{\text{DNE.}}$ □

See Maple for a picture of how a function can behave so counterintuitively.

Continuous Functions

Now that we have limits, we can define continuity of functions of several variables.

A function is continuous at a point provided it "behaves as expected" there.

Definition

We say that $f(x, y)$ is *continuous* at (a, b) provided

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b).$$

So if a function is known to be continuous at a certain point, we can evaluate the limit as we approach that point simply by evaluation.

What's Continuous?

Here's a partial list of common continuous functions:

- Polynomials (e.g. $x^3 + 3xy + y^2$) are continuous everywhere.
- Rational functions (e.g. $\frac{x^3+y^2}{2xy-5x+y^3}$) are continuous where they are defined.
- Compositions with continuous functions of one variable (e.g. $\sin(x^2 + y^2)$, e^{2x-3y}, $\arccos(2x) + 5y$) are continuous.
- Sums, products, and quotients (where defined) of continuous functions are continuous.

Examples

Example 7

Evaluate $\lim_{(x,y)\to(1,5)} \ln(x^2 + 2y - 3)$.

Solution. The polynomial $x^2 + 2y - 3$ is continuous everywhere.

Its value at $(1, 5)$ is $1^2 + 2 \cdot 5 - 3 = 8 > 0$, which falls in the domain of the logarithm.

Since the logarithm is continuous where it is defined, the composition $\ln(x^2 + 2y - 3)$ is continuous at $(1, 5)$.

Thus

$$\lim_{(x,y)\to(1,5)} \ln(x^2 + 2y - 3) = \ln(1^2 + 2 \cdot 5 - 3) = \boxed{\ln 8.}$$

Example 8

Where is the function

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

continuous?

Solution. The function $\dfrac{x^2 y}{x^2+y^2}$ is rational, so it is continuous everywhere it is defined: for $(x,y) \neq (0,0)$.

Therefore $f(x,y)$ is automatically continuous for $(x,y) \neq (0,0)$.

To test the continuity at $(0,0)$, we must let $(x,y) \to (0,0)$ and evaluate the limit.

Since $(x,y) \neq (0,0)$ when we take the limit, we have

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} = 0 = f(0,0),$$

by an earlier example.

It follows that $f(x,y)$ is continuous at $(0,0)$, too.

We conclude that $f(x,y)$ is continuous everywhere. $\square$

Remark. It's interesting to note that the function $\frac{x^2y}{x^2+y^2}$ has a removable discontinuity at the origin even though we cannot algebraically cancel the denominator. This never occurs for rational functions of one variable.