Partial Derivatives

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Calculus III

Introduction

Today we will begin studying the differential calculus of functions of several variables.

We will begin by considering *partial derivatives*.

As we will see, partial differentiation amounts to differentiating in each variable *separately*.

Our main tools will be the single variable differentiation rules from Calculus I.

Partial Derivatives

Definition

The *partial derivative* of $f(x,y)$ *with respect to* x is

$$f_x(x,y) = \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}.$$

The *partial derivative* of $f(x,y)$ *with respect to* y is

$$f_y(x,y) = \frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}.$$

Remarks.

- f_x is simply the "ordinary" derivative of f *treating* y *as a constant*. Likewise for f_y.
- We use subscripts instead of "prime" notation, and ∂ instead of d in the Leibniz notation.

Examples

In the following examples compute $f_x(x,y)$ and $f_y(x,y)$.

Example 1

$f(x,y) = x^4y^3 + 8x^2y$

Solution. Treating y as a constant we have

$$\boxed{f_x(x,y) = 4x^3y^3 + 16xy,}$$

and treating x as a constant we have

$$\boxed{f_y(x,y) = 3x^4y^2 + 8x^2.}$$

Example 2

$f(x, y) = x\cos(xy)$

Solution. If we hold y constant, we need the product rule and the chain rule to differentiate in x:

$$\boxed{f_x(x, y) = \cos(xy) - xy\sin(xy).}$$

On the other hand if we hold x constant, we only need the chain rule to differentiate in y:

$$\boxed{f_y(x, y) = -x^2\sin(xy).}$$

Example 3

$f(x, y) = x^y$

Solution. If we hold y constant, x^y is just a power function in x, so we can differentiate it with the power rule:

$$\boxed{f_x(x, y) = yx^{y-1}.}$$

But if we hold x constant, x^y is an *exponential* function in y (with base x). Thus

$$\boxed{f_y(x, y) = x^y \ln x.}$$

Remarks

The mechanics of differentiating in multiple variables are no different than those for single variable functions.

The primary difficulty in computing partial derivatives is simply keeping track of what is "variable" and what is "constant."

We can define partial derivatives in any number of variables in an analogous manner: we differentiate in one variable while holding the others fixed.

Examples

Example 4

If $g(x, y, z) = ze^{xyz}$, compute g_x, g_y and g_z.

Solution. Treating y and z as constants, the chain rule yields

$$\boxed{g_x(x, y, z) = yz^2e^{xyz}.}$$

Likewise, treating x and z as constants we find that

$$\boxed{g_y(x, y, z) = xz^2e^{xyz}.}$$

But if we treat x and y as constants, we need the product rule, too:

$$\boxed{g_z(x, y, z) = e^{xyz} + xyze^{xyz} = (1 + xyz)e^{xyz}.}$$

Example 5

If $h(x_1, x_2, \ldots, x_n) = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$, compute $\dfrac{\partial h}{\partial x_i}$.

Solution. Treating variable other that x_i as constant we find that

$$\frac{\partial h}{\partial x_i} = \frac{1}{2}\left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{-1/2} \cdot 2x_i$$

$$= \boxed{\frac{x_i}{\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}}}.$$

Understanding Partial Derivatives

Suppose we are given a function $f(x, y)$.

Given a constant k, the equation $y = k$ represents a vertical plane parallel to the xz-plane (perpendicular to the y-axis).

Recall that the graph of f is given by $z = f(x, y)$.

These two intersect where $z = f(x, k)$, which is a vertical cross section of the graph projected to the xz-plane.

The value of $f_x(x, k)$ represents the slope of the tangent line to this cross section.

As we vary k, the cross section changes, making f_x a function of both x and y.

Similarly, we can interpret f_y as the slope of the cross section $z = f(k, y)$ parallel to the yz-plane (perpendicular to the x-axis).

Put another way:

$$f_x(a,b) = \text{ ``slope'' of the graph of } f \text{ at the point } (a, b, f(a,b)) \text{ in the } x\text{-direction},$$

$$f_y(a,b) = \text{ ``slope'' of the graph of } f \text{ at the point } (a, b, f(a,b)) \text{ in the } y\text{-direction}.$$

More generally:

Partial derivative with respect to a variable = Rate of change *in the direction of that variable.*

Higher Order Partial Derivatives

Partial derivatives are functions of several variables, so have their own partial derivatives.

Definition

The *second order partial derivatives* of $f(x, y)$ are:

$$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2},$$

$$f_{yy} = (f_y)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2},$$

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x},$$

$$f_{yx} = (f_y)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}.$$

Remarks

- Subscript notation is read *left-to-right*.
- Leibniz notation is read *right-to-left*.
- f_{xx} and f_{yy} measure concavity of the graph of f in the x and y directions, respectively.
- The *mixed partial derivatives* f_{xy} and f_{yx} measure the tendency of the graph to "twist."
- We can talk about second order partial derivatives in any number of variables: in n variables there are n^2 second order partial derivatives.

Higher Order Derivatives

In general, if we differentiate $f(x, y)$ n times (with respect to any combination of x's and y's) we obtain an *nth order partial derivative*.

For example

$$f_{yxx} = (f_{yx})_x = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x \partial y}\right) = \frac{\partial^3 f}{\partial x^2 \partial y}$$

is a *third order mixed partial derivative*.

A function of two variables has 2^n (potentially distinct) derivatives of order n.

Examples

Find all second order partial derivatives of the following.

Example 6

$f(x, y) = x^4y^3 + 8x^2y.$

Solution. We have seen that

$$f_x(x, y) = 4x^3y^3 + 16xy \quad \text{and} \quad f_y(x, y) = 3x^4y^2 + 8x^2.$$

Taking the partial derivatives of f_x yields

$$f_{xx}(x, y) = 12x^2y^3 + 16y,$$
$$f_{xy}(x, y) = 12x^3y^2 + 16x.$$

Taking the partial derivatives of f_y yields

$$f_{yx}(x, y) = 12x^3y^2 + 16x,$$
$$f_{yy}(x, y) = 6x^4y.$$

Example 7

$f(x, y) = x\cos(xy)$.

Solution. We have seen that

$$f_x(x, y) = \cos(xy) - xy\sin(xy) \quad \text{and} \quad f_y(x, y) = -x^2\sin(xy).$$

Taking the partial derivatives of f_x yields

$$\begin{aligned} f_{xx}(x, y) &= -y\sin(xy) - y\sin(xy) - xy^2\cos(xy) \\ &= -2y\sin(xy) - xy^2\cos(xy), \\ f_{xy}(x, y) &= -x\sin(xy) - x\sin(xy) - x^2y\cos(xy) \\ &= -2x\sin(xy) - x^2y\cos(xy). \end{aligned}$$

Taking the partial derivatives of f_y yields

$$\begin{aligned} f_{yx}(x,y) &= -2x\sin(xy) - x^2 y\cos(xy), \\ f_{yy}(x,y) &= -x^3\cos(xy). \end{aligned}$$

Remarks.

- Notice that in both examples we have $f_{xy} = f_{yx}$.
- The somewhat amazing fact is that, under suitable hypotheses, this is usually true!

Equality of Mixed Partial Derivatives

Theorem 1 (Clairaut's Theorem)

If f_{xy} and f_{yx} are continuous, then $f_{xy} = f_{yx}$.

Remarks.

- This result is true for functions in any number of variables.
- It is also valid for higher order derivatives. For instance, provided they are all continuous, one must have

$$f_{xxy} = f_{xyx} = f_{yxx}.$$

- For the vast majority of the functions we will consider, Clairaut's theorem will hold automatically.

Example 8

If $u(r,\theta) = e^{r\theta}\sin\theta$, find $\dfrac{\partial^3 u}{\partial r^2 \partial\theta}$.

Solution. Strictly speaking, we are being asked to differentiate *first* in θ, then *twice* in r.

This requires us to use the product rule immediately, and then continue to differentiate in r, which becomes cumbersome.

Since the partial derivatives of u must all be continuous (why?), we can appeal to Clairaut's theorem and instead compute the two r derivatives first, which is easier.

Since $u(r,\theta) = e^{r\theta}\sin\theta$, we have

$$\begin{aligned}
u_r &= \theta e^{r\theta}\sin\theta, \\
u_{rr} &= \theta^2 e^{r\theta}\sin\theta, \\
u_{rr\theta} &= 2\theta e^{r\theta}\sin\theta + \theta^2 r e^{r\theta}\sin\theta + \theta^2 e^{r\theta}\cos\theta \\
&= \boxed{\theta e^{r\theta}(2\sin\theta + \theta r\sin\theta + \theta\cos\theta).}
\end{aligned}$$